

A STOCHASTIC FINITE-ELEMENT METHOD FOR TRANSFORMED NORMAL RANDOM PARAMETER FIELDS

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Transformed normal random fields are convenient models, e.g., for random material property fields obtained from microstructure analysis. In the context of the stochastic finite-element (FE) method, discretization of non-normal random fields by polynomial chaos expansions has been frequently employed. This introduces a non-linear relationship between the system matrix and normal random variables. For transformed normal random fields, the truncated Karhunen-Loève expansion of the transformed field is introduced into the stochastic FE formulation. This leads to a linear dependence of the system matrix on non-normal random variables. These non-normal random variables are then utilized to represent the discretized solution of the stochastic boundary value problem. Introduction of the approximations into the variational formulation of the stochastic boundary value problem and application of a collocation scheme yields a non-intrusive algorithm that allows coupling of reliability estimation procedures and existing FE solvers.

KEY WORDS: *polynomial chaos, stochastic finite-element method, stochastic sparse grid collocation, Smolyak algorithm, stochastic response surface method, adaptivity, random fields*

1. INTRODUCTION

Local properties of heterogeneous materials have been described in the past by transformed normal random fields (e.g., [1]). In contrast to normal random fields, non-normal fields may take into account the bounds of material properties and the asymmetry observed in histograms. Transformed normal random fields represent a large class of non-normal random fields that can be rather easily characterized due to the relation to an underlying normal random field.

In order to obtain local stress or displacement fields for structures, stochastic boundary value problems can be formulated by introducing random parameter fields into the differential equations of continuum mechanics. For the solution of these stochastic boundary value problems polynomial chaos expansions of response quantities have been widely utilized and are well documented [2–5]. In its original formulation in [2], a linear stochastic differential operator is discretized by a Karhunen-Loève expansion (KLE)—or a projection on the homogeneous chaos—and Hermite polynomials are introduced in order to represent the stochastic part of the solution, while finite-element (FE) discretizations are employed for the deterministic part. A Galerkin projection is then applied in order to determine the unknown expansion coefficients. This procedure has been extended to nonlinear problems e.g. in [4, 6].

For linear stochastic differential operators involving non-normal random fields, several authors propose a polynomial chaos representation of the random field and the solution [2, 4]. This approach has been extended in [7] to more general polynomial chaos families, belonging to the Askey scheme.

The theoretical foundation of Galerkin methods for stochastic elliptic partial differential equations has been laid in [4, 8, 9], where local and global polynomial chaos expansions were investigated and where a priori error estimates have been proved for a fixed number of terms of the KLE. In addition, solution methods for the algebraic problem were presented and analyzed in [10].

In recent years, several authors proposed stochastic collocation schemes instead of a Galerkin projection for the determination of quantities of interest [11–13]. Relations between the stochastic Galerkin method and the collocation approach have been clarified in [14]. A stochastic collocation scheme leads to non-intrusive algorithms, which allow computing the coefficients from repetitive runs of an FE solver for deterministic problems. If the dimension of the stochastic domain becomes high, sparse grid collocation procedures have been applied [15–17]. A non-intrusive algorithm based on least-squares regression has been presented in [18].

Recent developments include the application of polynomial chaos representations to the solution of stochastic systems; e.g., by introducing wavelet expansions [19] or local approximations [20], convergence acceleration by enrichment of the polynomial basis [21], reduction methods [22, 23] and multiscale approaches [24–26], to mention just a few.

Although the focus of most investigations concerning the approximate solution of stochastic boundary value problems have been on the computation of the first- and second-order moments of the solution, previously in [2] reliability computation techniques were described, which are based on series representations of the response distribution, the reliability index or Monte Carlo simulation techniques. In most of the papers that followed (e.g., [27, 28], and references therein), either the reliability index or simulation techniques have been employed. The approximate solution of the boundary value problem has been considered as a response surface. However, as has been pointed out in [29], a global approximation with Hermite polynomials might not be well suited to yield accurate higher-order moments or the distribution of the solution. Therefore, local approximations around the point of most probable failure have been introduced in [30]. In the context of the stochastic FE method, approximate solutions of the stochastic boundary value problem can be viewed as local stochastic response surfaces that depend on three quantities: the discretization level of the input random field, discretization parameter for the physical domain, and discretization parameter for the stochastic domain. Thus, a stochastic response surface can be viewed as a member of a three parameter family of metamodels. Alternatively, once the random field is discretized, sampling-based response surfaces [31, 32] could be considered as well.

In this paper, non-normal random fields that are transformations of normal random fields are considered. Three approaches to deal with stochastic boundary value problems involving transformations of non-normal random fields can be distinguished:

1. Simulation of the normal random field and application of the transformation in order to obtain samples of the non-normal random field.
2. Discretization of the non-normal random field by a (generalized) polynomial chaos representation.
3. KLE of the non-normal and the normal random field and explicit computation of the relationship between the non-normal and the normal random variables involved in the KLE.

Neither the first nor the second approach allow representing the non-normal random field by non-normal random variables, which could be advantageous (e.g., for sensitivity studies). The second approach might introduce significant approximation errors [29] and lead to a high number of expansion terms for the discretized random field.

On the other hand, the KLE is very efficient, because a linear dependence of the system matrix on the non-normal random variables is obtained. The eigenfunctions of the covariance kernel of the non-normal random field can be evaluated numerically. Unfortunately, the probability density function of the non-normal random variables is not available in analytical form, but samples of the non-normal random variables can be generated from samples of the underlying normal random variables. Therefore, application of a non-intrusive solution procedure based on a collocation scheme or a simulation-based algorithm make sense. In this investigation, collocation and simulation are connected to each other. In the stochastic domain, local approximations as well as global approximations based on a sparse grid interpolation scheme are introduced for the solution of the stochastic boundary value problem. For reliability assessment, a Monte Carlo simulation technique is introduced that makes use of the most probable point of failure (MPP), although other methods such as subset simulation or line sampling [33, 34] could be applied as well.

The paper is organized as follows. Section 2 introduces a stochastic linear elliptic boundary value problem as the model problem for structures with linear elastic but spatially random material properties. Section 3 discusses

the discretization of the transformed normal random field, Section 4 formulates the sparse collocation scheme, and Section 5 summarizes the proposed solution procedure. Section 6 briefly highlights reliability estimation methods. Finally, Section 7 presents two applications of the proposed method and in Section 8 conclusions are drawn.

2. STOCHASTIC LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

Let D be a convex bounded open set in \mathbb{R}^n and (Ω, \mathcal{F}, P) a complete probability space, where Ω is the set of outcomes, \mathcal{F} the σ field of events, and $P : \mathcal{F} \rightarrow [0 : 1]$ a probability measure. Consider the following model problem with stochastic operator and deterministic input function on $\bar{D} \times \Omega$: find $u : \bar{D} \times \Omega \rightarrow \mathbb{R}$, such that P almost surely:

$$-\nabla \cdot \alpha(x, \omega) \nabla u(x, \omega) = f(x) \text{ on } D, \quad u(x, \omega) = 0 \text{ on } \partial D. \quad (1)$$

It is assumed that the deterministic input function $f(x)$ is square integrable and that the random field $\alpha : D \times \Omega \rightarrow \mathbb{R}$ has a continuous and square-integrable covariance function

$$C(x, y) = \int_{\Omega} \{\alpha(x, \omega) - E[\alpha](x)\} \{\alpha(y, \omega) - E[\alpha](y)\} dP(\omega), \quad (2)$$

where the expectation operator $E[\alpha](x) = \int_{\Omega} \alpha(x, \omega) dP(\omega)$ denotes the mean value of the random field. Moreover, it is assumed that $\alpha(x, \omega)$ is bounded and coercive; i.e., there exists positive constants a_{\min}, a_{\max} , such that

$$P[\omega \in \Omega : a_{\min} < \alpha(x, \omega) < a_{\max} \forall x \in D] = 1. \quad (3)$$

We are interested in the probability that a functional $F(u)$ of the solution $u(x, \omega)$ exceeds a threshold F_0 ; i.e., we want to evaluate the integral

$$P_F = \int_{\Omega} \chi_{(F_0, \infty)} \{F[u(x, \omega)]\} dP(\omega), \quad (4)$$

where $\chi_I(\cdot)$, the indicator function, assumes the value 1 in the interval I and vanishes elsewhere.

The variational formulation of the stochastic boundary value problem necessitates the introduction of the Sobolev space $H_0^1(D)$ of functions having generalized derivatives in $L^2(D)$ and vanishing on the boundary ∂D with norm $\|u\|_{H_0^1(D)} = (\int_D |\nabla u|^2 dx)^{1/2}$, the space $L_P^2(\Omega)$ of square integrable random variables, and the tensor product space $H_0^1(D) \otimes L_P^2(\Omega)$ of $H_0^1(D)$ -valued random fields with finite second-order moments, equipped with the inner product

$$(u, v)_{H_0^1(D) \otimes L^2(\Omega)} = \int_{\Omega} \int_D \nabla u(x, \omega) \cdot \nabla v(x, \omega) dx dP(\omega). \quad (5)$$

The variational formulation of the stochastic linear elliptic boundary value problem [Eq. (1)] then reads: find $u \in H_0^1(D) \otimes L_P^2(\Omega)$, such that for all $v \in H_0^1(D) \otimes L_P^2(\Omega)$:

$$\int_{\Omega} \int_D \alpha(x, \omega) \nabla u \cdot \nabla v dx dP(\omega) = \int_{\Omega} \int_D f(x) v(x, \omega) dx dP(\omega). \quad (6)$$

The assumptions on the random field $\alpha(x, \omega)$ guarantee the continuity and coercivity of the bilinear form in Eq. (6) and, thus, the existence and uniqueness of a solution to Eq. (6) follows from the Lax-Milgram lemma.

The random field $\alpha(x, \omega)$ will now be replaced by a finite series of uncorrelated random variables. Due to the properties of the covariance function, the operator $T : L^2(D) \rightarrow L^2(D)$,

$$Tu = \int_D C(x, y) u(y) dy, \quad (7)$$

is compact and self-adjoint and, thus, admits a spectrum of decreasing non-negative eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$. The corresponding eigenfunctions $\{\phi_i(x)\}_{i=1}^{\infty}$ are orthonormal in $L^2(D)$. The random variables given by

$$\xi_i(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_D \{\alpha(x, \omega) - E[\alpha](x)\} \phi_i(x) dx \quad (8)$$

are uncorrelated (but, in general, not independent), have zero mean and unit variance and allow representing the random field by the KLE

$$\alpha(x, \omega) = E[\alpha](x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\omega) \phi_i(x), \quad (9)$$

that converges in $L^2(D \times \Omega)$ [35]. Conditions for stronger convergence properties are given in [9]. The KLE is usually truncated by retaining only the first M terms. This truncation is optimal in the sense that the mean-square error resulting from a finite representation of $\alpha(x, \omega)$ is minimized. In order to keep the computational effort small, a fast decay of the spectrum of Eq. (7) is important. It was shown in [36] that fast eigenvalue decay corresponds to smoothness of the covariance function.

3. DISCRETIZATION OF TRANSFORMED NORMAL RANDOM FIELDS

Consider now the case that $\alpha(x, \omega)$ is a transformed normal random field; i.e., $\alpha(x, \omega)$ is given by $\alpha(x, \omega) = T[g(x, \omega)]$, where $g(x, \omega)$ is a normal random field and $T(\cdot)$ a nonlinear transformation. Several methods to select $T(\cdot)$ have been discussed in the literature, cf. e.g. [1, 37].

In order to take advantage of the fact that the non-normal random field is a transformed normal random field, both the normal and the non-normal random field are discretized and the relationship between the normal and the non-normal random variables is computed. To this end, the truncated KLE $g_m(x, \omega)$ of the normal random field is introduced. However, the KLE of $T[g_m(x, \omega)]$ necessitates the computation of the covariance function of $T[g_m(x, \omega)]$, which might be cumbersome to compute. In the following, the covariance function of the transformed normal random field has been used instead.

The KLE of the normal random field $g(x, \omega)$ is given by

$$g(x, \omega) = E[g](x) + \sum_{i=1}^{\infty} \sqrt{\lambda_{Gi}} \xi_{Gi}(\omega) \phi_{Gi}(x). \quad (10)$$

where $g_m(x, \omega)$ denotes the truncation of Eq. (10) to order m . In the following, eigenvalues λ_{NGi} and eigenvectors ϕ_{NGi} are related to the operator

$$T_{NG}u = \int_D C_{NG}(x, y)u(x) dx, \quad (11)$$

where $C_{NG}(x, y)$ is the covariance function of the transformed normal field. The non-normal random field $\alpha(x, \omega) = T[g(x, \omega)]$ is approximated by

$$\alpha_M(x, \omega) = E[\alpha](x) + \sum_{i=1}^M \sqrt{\lambda_{NGi}} \xi_{NGi}(\omega) \phi_{NGi}(x), \quad (12)$$

where $\xi_{NGi}(\omega)$, $i = 1, 2, \dots, M$ are the non-normal random variables

$$\xi_{NGi}(\omega) = \frac{1}{\sqrt{\lambda_{NGi}}} \int_D \{T[g_m(x, \omega)] - E[T(g_m(x, \omega))]\} \phi_{NGi}(x) dx. \quad (13)$$

The series representation [Eq. (12)], being neither the KLE of $T[g(x, \omega)]$ nor of $T[g_m(x, \omega)]$, does not possess the error minimizing property of the KLE. The error can be represented as follows:

$$\begin{aligned} \|\alpha(x, \omega) - \alpha_M(x, \omega)\|_{L^2(D \times \Omega)} &\leq \|T[g(x, \omega)] - T[g_m(x, \omega)]\|_{L^2(D \times \Omega)} + \|T[g_m(x, \omega)] - E\{T[g_m(x, \omega)]\} \\ &- \sum_{i=1}^M \sqrt{\lambda_{mi}} \xi_{mi}(\omega) \phi_{mi}(x)\|_{L^2(D \times \Omega)} + \|E\{T[g_m(x, \omega)]\} - E[\alpha](x) + \sum_{i=1}^M \sqrt{\lambda_{mi}} \xi_{mi}(\omega) \phi_{mi}(x) \\ &- \sum_{i=1}^M \sqrt{\lambda_{NGi}} \xi_{NGi}(\omega) \phi_{NGi}(x)\|_{L^2(D \times \Omega)}, \end{aligned} \quad (14)$$

where

$$E\{T[g_m(x, \omega)]\} + \sum_{i=1}^M \sqrt{\lambda_{mi}} \xi_{mi}(\omega) \phi_{mi}(x) \quad (15)$$

is the truncated KLE of $T[g_m(x, \omega)]$. If the nonlinear transformation $T(\cdot)$ is Lipschitz continuous with constant C , one can estimate the first term as

$$\|T[g(x, \omega)] - T[g_m(x, \omega)]\|_{L^2(D \times \Omega)} \leq C \|g(x, \omega) - g_m(x, \omega)\|_{L^2(D \times \Omega)}, \quad (16)$$

which converges for $m \rightarrow \infty$, because $g_m(x, \omega)$ is the truncated KLE of $g(x, \omega)$. The second term contains the difference of $T[g_m(x, \omega)]$ and its KLE and converges for $M \rightarrow \infty$. For the last term, one has

$$\begin{aligned} & \|E\{T[g_m(x, \omega)]\} - E[\alpha](x) + \sum_{i=1}^M \sqrt{\lambda_{mi}} \xi_{mi}(\omega) \phi_{mi}(x) - \sum_{i=1}^M \sqrt{\lambda_{NGi}} \xi_{NGi}(\omega) \phi_{NGi}(x)\|_{L^2(D \times \Omega)} \\ &= \|E\{T[g_m(x, \omega)] - \alpha(x, \omega)\} + \sum_{i=1}^M \int_D \{T[g_m(y, \omega)] - E[T(g_m(y, \omega))]\} \\ &\quad \times [\phi_{mi}(x) \phi_{mi}(y) - \phi_{NGi}(x) \phi_{NGi}(y)] dy\|_{L^2(D \times \Omega)} \end{aligned} \quad (17)$$

and this term will remain small, if the eigenfunctions of the operator [Eq. (7)] with the covariance function of $T[g_m(x, \omega)]$ and $T[g(x, \omega)]$ do not differ too much.

The random variables $\xi_{NGi}(\omega)$, $i = 1, 2, \dots, M$ are neither independent nor uncorrelated.

4. SPARSE COLLOCATION SCHEME

The series representation [Eq. (12)] is inserted into Eq. (1) to yield

$$-\nabla \cdot \{E[\alpha](x) + \sum_{i=1}^M \sqrt{\lambda_{NGi}} \xi_{NGi}(\omega) \phi_{NGi}(x)\} \nabla u(x, \omega) = f(x) \text{ on } D. \quad (18)$$

It is assumed that $\Gamma_i = \xi_{NGi}(\Omega)$ are bounded intervals in \mathbb{R} . Denote with Γ the Cartesian product $\Gamma = \prod_{i=1}^M \Gamma_i \subset \mathbb{R}^M$. Making use of the fact that $\alpha(x, \omega)$ has been represented by a finite number of random variables, an auxiliary problem is considered in the following:

$$-\nabla \cdot \{E[\alpha](x) + \sum_{i=1}^M \sqrt{\lambda_{NGi}} y_i \phi_{NGi}(x)\} \nabla u(x, y) = f(x) \text{ for } (x, y) \in D \times \Gamma. \quad (19)$$

From the expression for $u(x, y)$, one obtains $u(x, \omega)$ by replacing the vector y with the vector of random variables $\xi_{NGi}(\omega)$, $i = 1, 2, \dots, M$.

For the physical domain, this equation is discretized on finite-dimensional approximation spaces. For $H_0^1(D)$, a family of standard FE approximation spaces $X_h \subset H_0^1(D)$ of continuous piecewise linear functions in a regular triangulation \mathcal{T}_h of D with mesh parameter h is considered.

Denote with $N_i(x)$, $i = 1, 2, \dots, N$, a basis of $X_h \subset H_0^1(D)$. The solution $u(x, y)$ is approximated by

$$u(x, y) = \sum_{i=1}^N u_i(y) N_i(x). \quad (20)$$

For a fixed value y , the unknown coefficients $u_i(y)$, $i = 1, 2, \dots, N$ can be computed from the solution of the FE problem

$$\left(K^{(0)} + \sum_{s=1}^M K^{(s)} y_s \right) \mathbf{u}(y) = \mathbf{f}, \quad (21)$$

where the matrices $\mathbf{K}^{(0)}$, $\mathbf{K}^{(s)}$, $s = 1, 2, \dots, M$ and vector \mathbf{f} are given by

$$\begin{aligned} K_{ij}^{(0)} &= \int_D E[\alpha](x) \nabla N_i(x) \cdot \nabla N_j(x) \, dx, \\ K_{ij}^{(s)} &= \sqrt{\lambda_{NGs}} \int_D \Phi_{NGs}(x) \nabla N_i(x) \cdot \nabla N_j(x) \, dx, \\ f_i &= \int_D f(x) N_i(x) \, dx, \quad i, j = 1, 2, \dots, N \end{aligned} \quad (22)$$

and $\mathbf{u}(y)$ is the vector containing the nodal displacements $u_i(y)$, $i = 1, 2, \dots, N$ for the fixed value y . The matrices $\mathbf{K}^{(s)}$, $s = 1, 2, \dots, M$ can be interpreted as FE stiffness matrices with a spatial variation of Young's modulus.

As the distribution of the random variables $\xi_{NGi}(\omega)$, $i = 1, \dots, M$ is difficult to obtain, we seek to compute $\mathbf{u}(y)$ at given nodes in the stochastic domain to which Lagrange interpolation can be applied. Samples of the random variables $\xi_{NGi}(\omega)$, $i = 1, \dots, M$ can then be generated and inserted into the interpolation polynomials in order to obtain samples of the solution.

The stochastic domain Γ is partitioned into non-overlapping elements $\gamma = \prod_{i=1}^M [a_i^\gamma, b_i^\gamma]$. On each element γ , the approximation space is the span of tensor product polynomials with degree at most $q = (q_1, q_2, \dots, q_M)$. The approximation subspace $P_{q,k}(\Gamma) \subset L_p^2(\Gamma)$ comprises thus the FE space of possibly discontinuous piecewise polynomials with degree at most q and mesh parameter $k = (k_1, k_2, \dots, k_M)$, with $k_i = \max_{\gamma \subset \Gamma} |a_i^\gamma - b_i^\gamma|$. If the partition consists of a single element only and the order q is varied, global approximations are obtained.

On each FE $\gamma \subset \Gamma$, the solution is constructed by Lagrange interpolation on a suitable set of points $\eta^q = \{y_i, i = 1, 2, \dots, m_q\}$, where q indicates the interpolation order. The interpolation to order q requires m_q values. Considering first the case of a single stochastic dimension, $\mathbf{u}(y)$ is approximated on each element $\gamma \subset \Gamma$ by a sequence of Lagrange interpolation operators

$$U^q(\mathbf{u})(y) = \sum_{j=1}^{m_q} \mathbf{u}(y_j) l_j(y), \quad (23)$$

where $\mathbf{u}(y)$ must be an element of $C^0[\gamma, H_0^1(D)]$. In the multi-dimensional case, a tensor product interpolation

$$(U_1^q \otimes U_2^q \otimes \dots \otimes U_M^q)(\mathbf{u})(y) = \sum_{j_1=1}^{m_{q_1}} \sum_{j_2=1}^{m_{q_2}} \dots \sum_{j_M=1}^{m_{q_M}} \mathbf{u}(\eta_{j_1}^{q_1}, \eta_{j_2}^{q_2}, \dots, \eta_{j_M}^{q_M}) (l_{j_1}^{q_1} \otimes l_{j_2}^{q_2} \otimes \dots \otimes l_{j_M}^{q_M})(y), \quad (24)$$

where $\eta_{j_k}^{q_k}$ is the j_k th point of the set η^{q_k} and $l_{j_k}^{q_k}$ is the j_k th Lagrange polynomial for interpolation until the order q_k is practically not feasible. Therefore, interpolation on sparse sets of support nodes by means of Smolyak's algorithm [38] is introduced. With the nested interpolants $\Delta^i = U^i - U^{i-1}$, the Smolyak interpolant of $\mathbf{u}(y)$ is

$$\begin{aligned} A_{q,M}(\mathbf{u}) &= \sum_{i \in X_{q,M}} (\Delta^{i_1} \otimes \Delta^{i_2} \otimes \dots \otimes \Delta^{i_M})(\mathbf{u}) \\ &= A_{q-1,M}(\mathbf{u}) + \sum_{|i|=q} (\Delta^{i_1} \otimes \Delta^{i_2} \otimes \dots \otimes \Delta^{i_M})(\mathbf{u}) \\ &= \sum_{i \in Y_{q,M}} (-1)^{q+M-|i|} \binom{M-1}{q+M-|i|} (U^{i_1} \otimes U^{i_2} \otimes \dots \otimes U^{i_M})(\mathbf{u}), \end{aligned} \quad (25)$$

where

$$\begin{aligned} X_{q,M} &= \left\{ i \in \mathbb{N}_+^M, i \geq 1 : \sum_{j=1}^M (i_j - 1) \leq q \right\}, \\ Y_{q,M} &= \left\{ i \in \mathbb{N}_+^M, i \geq 1 : q - M + 1 \leq \sum_{j=1}^M (i_j - 1) \leq q \right\}, \end{aligned} \quad (26)$$

and $|i| = i_1 + i_2 + \dots + i_M$. To compute the Smolyak interpolant, one only needs to evaluate function $\mathbf{u}(y)$ on the sparse grid

$$\bigcup_{i \in Y_q^M} \eta^{i_1} \times \eta^{i_2} \times \dots \times \eta^{i_M}. \quad (27)$$

By mapping the element $\gamma \subset \Gamma$ on the cube $[0, 1]^M$ and choosing appropriate interpolation points, e.g., the Clenshaw-Curtis grid points with equidistant nodes on $[0, 1]$ [39], given by

$$\begin{aligned} \eta^1 &= \{0.5\}, \\ \eta^i &= \left\{ \frac{j-1}{m_i-1}, j = 1, 2, \dots, m_i = 2^{i-1} + 1 \right\}, i > 1; \end{aligned} \quad (28)$$

or the extrema of Chebyshev polynomials or Gauss-Lobatto points, one obtains nested sets of interpolation points $\eta^i \subset \eta^{i+1}$ and, thus, the set of evaluation points reduces further. In order to extend the interpolation by one level, it is only necessary to evaluate the function at the grid points

$$\bigcup_{|i|=q} \eta_{\Delta}^{i_1} \times \eta_{\Delta}^{i_2} \times \dots \times \eta_{\Delta}^{i_M}, \quad (29)$$

where $\eta_{\Delta}^i = \eta^i \setminus \eta^{i-1}$.

Figure 1 displays the set of interpolation points for $q = 7$ and $M = 3$. It comprises 177 nodes.

5. LOCAL RESPONSE SURFACE

Once the algebraic problems are solved and the approximation coefficients have been determined, an expression of the displacement field that depends on the input random variables has been obtained. This expression can be considered as a response surface. This response surface has local character [30], and depends on the size and location of the elements γ , if a partition of Γ is adopted.

The approximation error for $u(x, \omega)$ is the sum of three parts:

- The truncation error of the input random field, controlled by the expansion order m and M .
- The FE discretization error, controlled by the mesh parameter h .

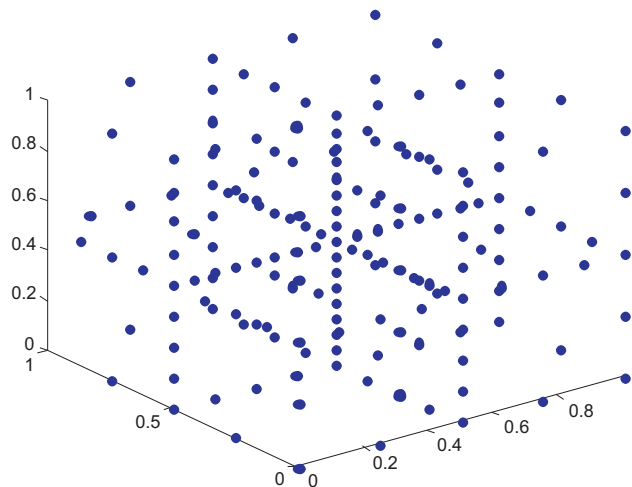


FIG. 1: Clenshaw-Curtis grid points: $M = 3$, $q = 7$.

- The discretization error on the stochastic domain, controlled by the mesh parameter k and the interpolation order q .

As can be seen, the parallelization of the solution procedure is easily possible due to the fact that:

- The approximations on each element $\gamma \subset \Gamma$ are independent.
- For each element γ , the evaluations at the interpolation points are independent.

As a consequence, any degree of parallelization (from coarse grained to fine grained) can be applied, depending on the number of processors at disposal.

6. RELIABILITY ASSESSMENT

Computation of approximations for the distribution of $u(x, \omega)$ or the failure probability Eq. (4) is a complex task and resorting to Monte Carlo simulation via the obtained expression for $u(x, \omega)$ seems to be the easiest way to accomplish it.

For solving reliability problems, it is very helpful to consider the approximation for the displacements as a local response surface, which leads to a functional relationship between the input random variables and $u(x, \omega)$. Due to the relationship between the non-normal random variables $\xi_{NGi}(\omega)$ and the standard normal variables of the underlying KLE of the normal random field, it is then possible to directly compute the MPP (i.e., the point in standard normal space with the lowest Euclidean norm that satisfies $F\{u[x, \xi_{NG1}(\omega), \xi_{NG2}(\omega), \dots, \xi_{NGM}(\omega)]\} = F_0$) and to adapt the partition of Γ at the vicinity of the corresponding values for the non-normal random variables. In this way, the reliability problem is solved with a high degree of accuracy in an adaptive manner.

The MPP may also be useful for the evaluation of the integral in Eq. (4) by means of variance-reduced Monte Carlo simulation (importance sampling). To this end, a sampling density $\tilde{p}(z)$ is introduced by shifting probability density function $p(z)$ of the standard normal random variables to the previously obtained MPP, and Eq. (4) is approximated by

$$P_F \approx \sum_{j=1}^N \chi_{F_0, \infty} \{F[u(x, z^j)]\} \frac{p(z^j)}{\tilde{p}(z^j)} \tilde{p}(z^j), \quad (30)$$

where the sampling points z^j , $j = 1, 2, \dots, N$ are generated according to $\tilde{p}(z)$, and $u(x, z^j)$ is computed from the approximation on the element $\gamma \subset \Gamma$ that contains $y(z^j)$.

The preceding discussion has highlighted the importance of the MPP for reliability assessment. This procedure can be applied if the number of random variables involved in the problem is relatively small. If this is the case, the Euclidean distance from the origin to the MPP in standard normal space can be used as a quality measure to adapt the parameters of the numerical solution procedure, see [30]. A variant of this procedure works with a fixed ratio between m and M .

7. EXAMPLES

The first example deals with a standard problem for stochastic FE techniques, a clamped thin square plate under uniform in-plane tension q , cf. [2]. The problem is depicted in Fig. 2. The product of Young's modulus and the thickness of the plate is assumed to be an isotropic non-normal random field with covariance function

$$C(x_1, y_1; x_2, y_2) = \sigma^2 \exp \left(-\frac{|x_1 - x_2|}{l_c} - \frac{|y_1 - y_2|}{l_c} \right), \quad (31)$$

with standard deviation $\sigma = 0.2$, correlation length $l_c = 1$, and unit mean value. The relation between non-normal random field $\alpha(x, \omega)$ and normal random field $g(x, \omega)$ is given by $\alpha(x, \omega) = g^2(x, \omega)$. Poisson's ratio is set to 0.3.

The plate has unit length l . A threshold value is assumed for the maximum longitudinal displacement and failure occurs if the maximum longitudinal displacement exceeds this threshold value.

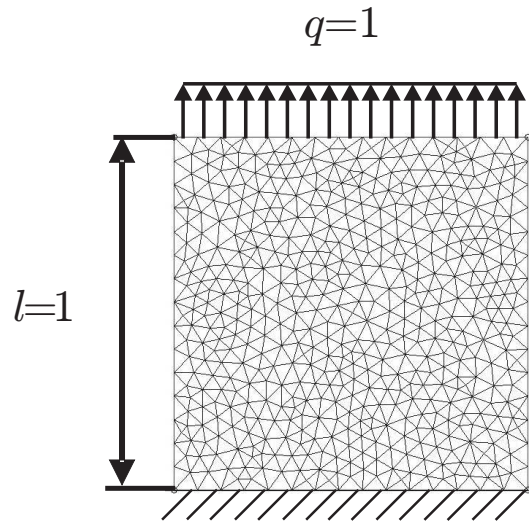


FIG. 2: Thin square plate under uniform in-plane tension.

The non-normal random variables of the KLE of the non-normal random field have been computed according to Eq. (13). The histograms are given in Fig. 3. The distributions of the non-normal random variables exhibit a leptokurtic behavior.

For a maximum displacement of 1.19 and a discretization of the random fields with $M = 6$ and $m = 3$ terms, Fig. 4 displays the development of the relative errors for the failure probability when the interpolation order is increased. In order to compare the results, the abscissa displays the number of deterministic FE runs. The exact value of the failure probability has been estimated from importance sampling with 50,000 FE runs without recurrence to a response surface.

In order to obtain local approximations, an interval of length σ has been inserted at the point corresponding to the MPP for the first and for the first two coordinates with the largest partial derivative of the longitudinal displacement at

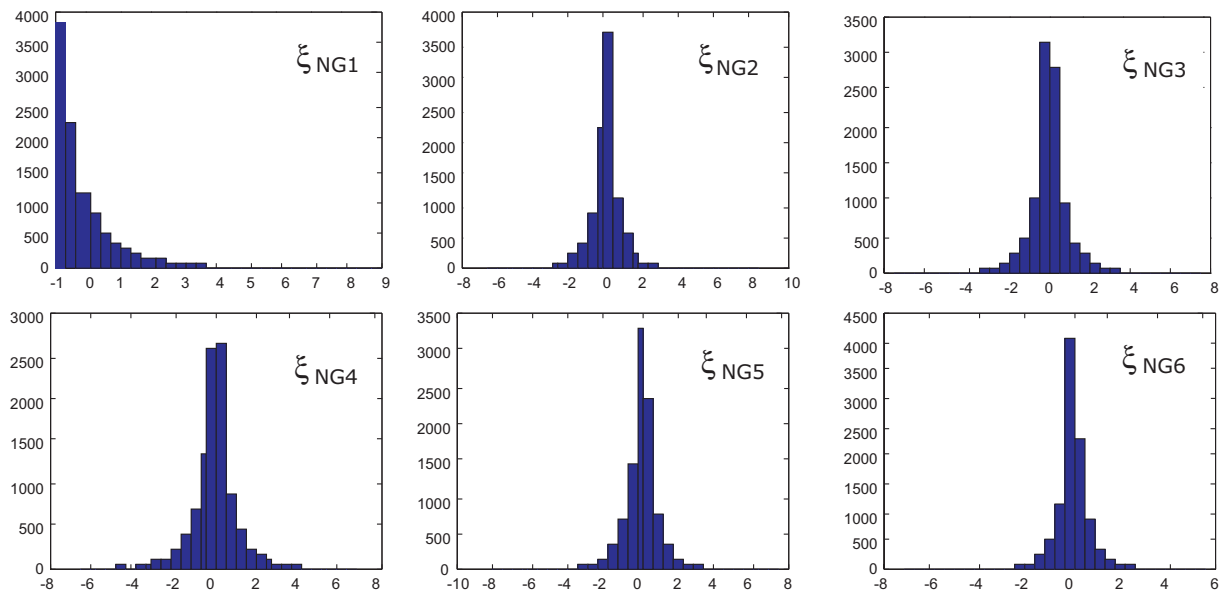


FIG. 3: Histograms of the first six non-normal random variables.

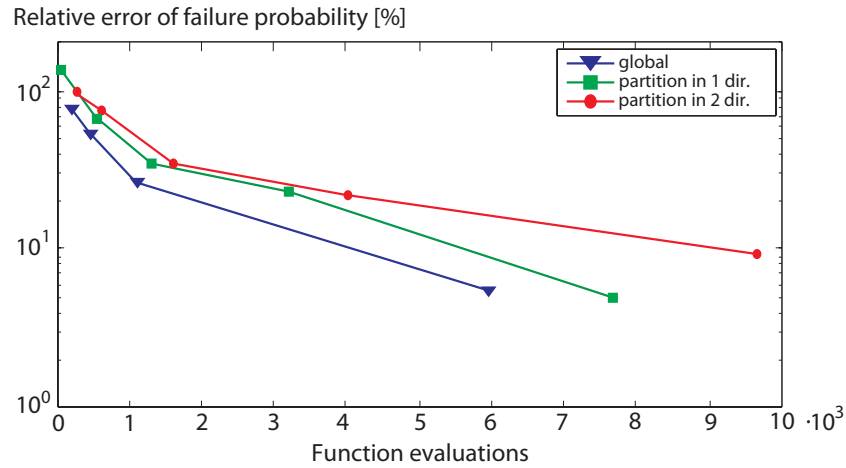


FIG. 4: Percentage relative error of the computed failure probability versus number of function evaluations obtained with global and local approximation methods.

that point, respectively. Figure 4 compares global and local approximations with respect to function evaluations (i.e., deterministic FE runs), where the interpolation support nodes are the Clenshaw-Curtis grid points. For the case under consideration, the results indicate that global approximations are more efficient than local approximations. This result is in contrast to previous findings by the author for truncated normal random fields, cf. [30], and could be due to the special partition of the stochastic domain or the design of the stochastic sparse grid that is applied in each element.

The second example deals with the embankment dam problem considered in [40]. It will be shown that no additional expansion coefficients for the non-normal random field discretization are necessary. The system is depicted in Fig. 5. It consists of an embankment dam made of heterogeneous material with trapezoidal cross section under compression by deterministic loads ($q = 30$ kPa, mass density of the dam 1800 kg/m³). The parameters are the same as for Case 1 in [40]. Especially, the elastic and shear moduli $E(x, z)$ and $G(x, z)$ are described by non-homogeneous lognormal random fields:

$$E(x, z) = m_E(z) + \sigma_E \frac{\exp[Y(x, z)] - m_Y}{\sigma_Y}, \quad G(x, z) = m_G(z) + \sigma_G \frac{\exp[Y(x, z)] - m_Y}{\sigma_Y}, \quad (32)$$

with $m_E(z) = 30 - 0.5z$, $\sigma_E = 12.5$, $m_G(z) = 12 - 0.1z$, $\sigma_G = 5$, $m_Y = \exp(0.5)$, $\sigma_Y^2 = \exp(2) - \exp(1)$, and a zero-mean, unit variance normal random field $Y(x, z)$ with autocorrelation function [40]

$$R(\Delta x, \Delta z) = \exp(-|\Delta x|/10) \exp(-|\Delta z|/3). \quad (33)$$

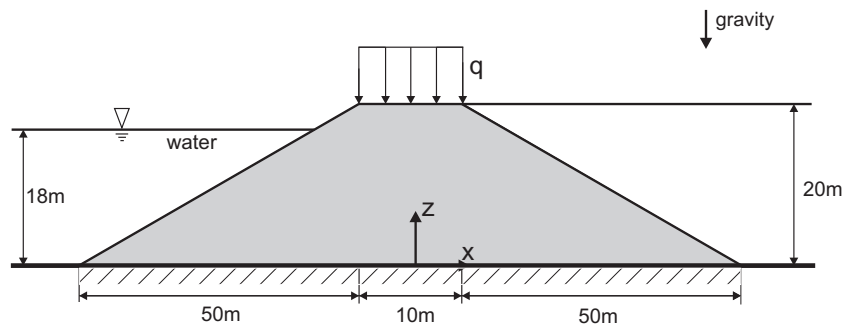


FIG. 5: Sketch of the embankment dam problem.

All coordinates are measured in meters. The physical domain is discretized into 2160 three-noded plane strain finite elements. Failure is assumed to occur if the Mohr-Coulomb criterion

$$\tau \geq c + \sigma_n \tan \phi \quad (34)$$

is satisfied in at least one element, where τ is the shear stress, σ_n the normal stress, $c = 125$ kPa the cohesive strength and $\phi = 30$ the friction angle.

As the correlation length compared to the dimensions of the embankment dam is rather small, a high number of terms in the KLE have to be considered, cf. also [41]. The large number of random variables renders the determination of a design point difficult.

Figure 6 depicts the development of the approximations for the failure probability with increasing number of KLE terms. The results were obtained by direct Monte Carlo simulation. For KL-G, the Gaussian random field has been represented by its m term KLE and then the transformation by the exponential function has been applied. For KL-NG, non-normally distributed random variables were generated (with $m = M$ for the underlying normal random variables) according to the proposed procedure. It can be seen that approximately 60 KLE coefficients are necessary in order to approximate the reference value for the failure probability well, but that no additional expansion coefficients are necessary in the discretization of the non-normal random field. This is certainly due to the transformation of the standard normal random field by the exponential function. In contrast, a second-order polynomial chaos representation of the non-normal random field would have to deal with 1891 expansion terms [41].

8. CONCLUSIONS

In this paper, a stochastic FE method for transformed normal random fields is proposed. Unlike other methods, it relies fully on the KLE of the non-normal random field and avoids polynomial chaos representations of the non-normal field. The proposed method necessitates the introduction of a collocation scheme for the stochastic domain. Recurrence to sparse grid interpolation methods for this task helps to keep the computational effort small.

Although discretization of the stochastic domain by discontinuous FEs has been discussed, the example under consideration indicates that global approximations might be a more efficient choice in terms of deterministic FE runs. Application of discontinuous FEs could be improved by further optimizing the partition of the stochastic domain and the selection of the interpolation nodes.

Finally, it is noted that sparse grid interpolation also could be directly applied to the normal random variables of the KLE for the underlying normal random field. However, the application of the interpolation to the non-normal random variables is a more direct alternative and the computation of the non-normal random variables from the normal random variables is rather inexpensive.

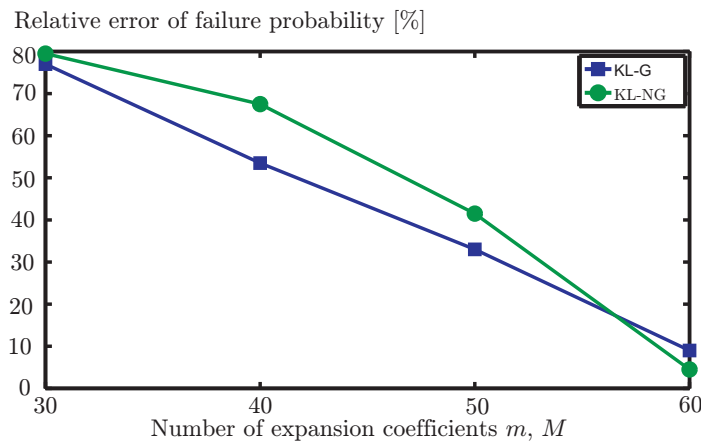


FIG. 6: Percentage relative error of the computed failure probability versus number of function evaluations obtained with global and local approximation methods.

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