International Journal for Uncertainty Quantification, 1 (4): 351–367 (2011)

NUMERICAL SOLUTIONS FOR FORWARD BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS AND ZAKAI EQUATIONS

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Original Manuscript Submitted: 05/22/2011; Final Draft Received: 08/24/2011

The numerical solutions of decoupled forward backward doubly stochastic differential equations and the related stochastic partial differential equations (Zakai equations) are considered. Numerical algorithms are constructed using reference equations. Rate of convergence is obtained through rigorous error analysis. Numerical experiments are carried out to verify the rate of convergence results and to demonstrate the efficiency of the proposed numerical algorithms.

KEY WORDS: *SPDEs, Zakai equation, backward SDEs, forward backward doubly stochastic differential equations, conditional expectation*

1. INTRODUCTION

In this paper, we consider the numerical solution of forward backward doubly stochastic differential equations (FBDSDEs)

$$\begin{cases} X_s^{t,x} &= x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \quad t \le s \le T \\ Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})dr + \int_s^T g(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})d\overleftarrow{B}_r \\ & \quad - \int_s^T Z_r^{t,x} dW_r, \quad t \le s \le T \end{cases} \tag{1}$$

in relation to the following stochastic parabolic partial differential equation in backward form:

$$u_t(x) = h(x) + \int_t^T \{\mathcal{L}u_s(x) + f[s, x, u_s(x), \nabla u_s(x)]\}ds + \int_t^T g[s, x, u_s(x),\ \nabla u_s(x)]d\overleftarrow{B}_s,\ t \in [0, T] \tag{2}$$

Here f and g are given functions, B_t and W_t, $0 \le t \le T$, are independent standard Brownian motions, the integrals involved with $d\overleftarrow{B}_t$ are the Itô integrals with backward integration whose meaning will be made clear in Section 2, $\mathcal{L}$ is the elliptic partial differential operator defined as

$$(\mathcal{L}u_t)_i(x) := [L(u_t)_i](x), \quad 1 \le i \le k$$

with

$$L := \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$$

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2152–5080/11/$35.00

Equation (2) is also called the Zakai equation, which can be derived as the density function of the conditional expectation in nonlinear filter problems [1].

There have been extensive studies on numerical solutions of Zakai equations, both as general stochastic parabolic partial differential equations and in the context of nonlinear filter problems. Several methodologies have been developed, among them are the finite difference methods based on the discretization of general stochastic parabolic partial differential equations [2–6], Wang-Zakai approximations [7], spectral methods based on Wiener chaos expansions of the exact solutions [8, 9], and the Monte Carlo method based on particle approximation of the conditional expectation [10–12]. We also Refs. [13] and [14] and references therein for more recent developments.

In this paper, we attempt to construct an efficient numerical algorithm for the FBDSDE (1) and to solve the Zakai equation (2) numerically through an equivalent relationship between (1) and (2). Such an equivalent relationship was proved by Pardoux and Peng [15, 16]. An FBDSDE is the extension of a backward forward stochastic differential equation (FBSDE), which is the same as (1) without the Brownian motion B involved. There have also been numerous attempts to solve FBSDEs numerically. The first such an attempt was due to Ma et al. [17] where a four-step scheme was constructed through numerical approximation of a related partial differential equation. A binomial tree approach was proposed in [18] and further developed later in [19]. In both aforementioned approaches, high regularity is required for the input data and/or the exact solution. A direct numerical approximation was developed in [20] which reduced the regularity requirements. More recently, a class of high-order numerical algorithms is constructed through reference equations and a multistep method [21, 22]. In this research, we follow the general framework of [21] to construct efficient numerical algorithms for (1). The novelty of our approach is threefold. as follows:

1. Though there have been extensive studies on the theory and applications of FBDSDEs, to the best of our knowledge, this is the first attempt to obtain practical algorithms to solve FBDSDEs numerically.

2. Our algorithm provides numerical solutions for both the solution u of (2) and its gradient ∇u. This is significant because, in many practical problems, the diffusion flux is a more useful physical quantity.

3. Our numerical algorithm enjoys the same convergence rate as other finite difference algorithms for solving parabolic SPDEs [3, 6]; however, the error analysis of our algorithm is less complicated. Furthermore, our algorithm has the potential to be extended to obtain higher order schemes.

The rest of the paper is organized as follows. In Section 2, we give a brief introduction to BDSDEs and their relations to SPDEs. In Section 3, we propose a class of numerical algorithms for BDSDE (1) and derive their error estimates. Then in Section 4, we present numerical experiments to verify the rate of convergence results derived in Section 3. A few concluding remarks are given in Section 5.

2. BDSDES AND SPDES

To derive the numerical algorithm and conduct its convergence analysis, we provide a brief introduction to FBDSDEs and the relationship between FBDSDEs and the SPDE (2).

Let $(\Lambda, \mathcal{F}, P)$ be a complete probability space and $T > 0$ be the terminal time, $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two mutually independent standard Brownian motions defined on $(\Lambda, \mathcal{F}, P)$ with their values in $\mathbb{R}^d$ and in $\mathbb{R}^l$, respectively. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

where $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}$ the σ-field generated by $\{\eta_r - \eta_s; s \leq r \leq t\}$, and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$ for a stochastic process η. Note that the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing and is not a filtration. For positive integer $n \in \mathbb{N}$, we define spaces $M^2(0, T; \mathbb{R}^n)$ and $S^2([0, T]; \mathbb{R}^n)$ as follows.

$$M^2(0, T; \mathbb{R}^n) := \{\varphi_t | \varphi_t \in \mathbb{R}^n, E\int_0^T |\varphi_t|^2 dt < \infty, \ \ \varphi_t \in \mathcal{F}_t, \ \text{ a.e. } t \in [0, T]\}$$

and

$$S^2([0,T];\mathbb{R}^n) := \{\varphi_t | \varphi_t \in \mathbb{R}^n, E(\sup_{0\le t\le T} |\varphi_t|^2) < \infty,\ \varphi_t \in \mathcal{F}_t,\ t \in [0,T]\}$$

Let

$$f : \Lambda \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$$

and

$$g : \Lambda \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}$$

be jointly measurable such that for any $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$f_t(y,z) \in M^2(0,T;\mathbb{R}^k)$$

$$g_t(y,z) \in M^2(0,T;\mathbb{R}^{k\times l})$$

We assume, moreover, that there exist constants $c > 0$ and $0 < \alpha < 1$ such that for any $(\omega,t) \in \Lambda \times [0,T]$, (y_1,z_1), $(y_2,z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times l}$,

$$|f_t(y_1,z_1) - f_t(y_2,z_2)|^2 \le c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2)$$

$$\|g_t(y_1,z_1) - g_t(y_2,z_2)\|^2 \le c|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2$$

From [16], under the above assumptions and standard conditions on b and σ, we know that there exists a pair of processes $\{(Y_s^{t,x}, Z_s^{t,x}); (t,x) \in [0,T]\times\mathbb{R}^d\}$ that is the unique solution to the following FBDSDE: For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \quad t \le s \le T \tag{3}$$

$$Y_s^{t,x} + h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})dr + \int_s^T g(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})d\overleftarrow{B}_r - \int_s^T Z_r^{t,x}dW_r,\ t \le s \le T \tag{4}$$

where $(Y_s^{t,x}, Z_s^{t,x}) \in S^2([0,T];\mathbb{R}^k) \times M^2(0,T;\mathbb{R}^{k\times l})$. Here, $d\overleftarrow{B}_r$ denotes the backward Itô integration, i.e., for a $\mathcal{F}_{t,T}^B$ adapted process V_t, and quasi-uniform time partitions Δ: $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$,

$$\int_0^T V_t d\overleftarrow{B}_t := \lim_{\Delta t\to 0} \sum_{n=0}^N V_{t_{n+1}}(B_{t_{n+1}} - B_{t_n})$$

where $\Delta t = \max_{0\le i\le N-1}(t_{i+1} - t_i)$. According to Pardoux and Peng [16], we have the following nonlinear Feynman-Kac formula:

$$Y_s^{t,x} = u_s(X_s^{t,x}),\ Z_s^{t,x} = (\nabla u_s \sigma)(X_s^{t,x});\ (t,x) \in [0,T]\times\mathbb{R}^d$$

where $u = u_t(x) \in R^k$ is the unique solution of the following system of the backward stochastic partial differential equation:

$$u_t(x) + h(x) + \int_t^T \{\mathcal{L}_s u_s(x) + f[s,x,u_s(x),(\nabla u_s\sigma)(x)]\}ds + \int_t^T g[s,x,u_s(x),\ (\nabla u_s\sigma)(x)]d\overleftarrow{B}_s,\ 0 \le t \le T \tag{5}$$

To simplify our presentation, in what follows we assume that $b \equiv 0$ and $\sigma \equiv 1$ in Eq. (3). Thus we have

$$X_s^{0,x} = x + W_s, \quad x \in \Omega, s \in [0,T]$$

and the elliptic partial differential operator L becomes

$$L = \frac{1}{2}\sum_i^d \frac{\partial^2}{\partial x_i^2}$$

When the spatial domain Ω of the SPDE (5) is a subset of $\mathbb{R}^d$ and the boundary condition

$$u_t(x) = \gamma(t, x), \quad \text{on} \quad [0, T] \times \partial\Omega$$

is prescribed, the corresponding BDSDE is defined through a stopping time τ defined as

$$\tau = \inf\{s; X_s^{t,x} \in \partial\Omega, s \geq t, x \in \Omega\} \tag{6}$$

Then, we have the BDSDE with stopping time as follows:

$$\begin{aligned} Y_s^{t,x} \;&=\; \Phi(X_{\tau\wedge T}^{x}) + \int_s^{T\wedge\tau} f(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})dr + \int_s^{T\wedge\tau} g(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})d\overleftarrow{B}_r \\ &-\; \int_s^{T\wedge\tau} Z_r^{t,x} dW_r, \quad t \leq s \leq T, \; x \in \Omega \end{aligned} \tag{7}$$

where $\Phi(X_{\tau,T}^{x}) = h(X_T^{t,x})I_{\tau\geq T} + \gamma(\tau, X_\tau^{t,x})I_{\tau\leq T}$. When $t = 0$, the stopping time τ defined in Eq. (6) becomes

$$\tau = \inf\{s; X_s^{0,x} \in \partial\Omega, \; s \geq 0 \, x \in \Omega\}$$

Thus, BDSDE (7) changes to the following equation:

$$\begin{aligned} Y_t^{0,x} \;&=\; \Phi(X_{\tau\wedge T}^{0,x}) + \int_t^{T\wedge\tau} f(s, X_s^{0,x}, Y_s^{0,x}, Z_s^{0,x})ds - \int_t^{T\wedge\tau} Z_s^{0,x} dW_s \\ &+\; \int_t^{T\wedge\tau} g(s, X_s^{0,x}, Y_s^{0,x}, Z_s^{0,x})d\overleftarrow{B}_s, \; t \in [0, T], \quad x \in \Omega \end{aligned}$$

where for given x, $X_0^{0,x} = x$, $\Phi(X_{\tau\wedge T}^{0,x}) = h(X_T^{0,x})I_{\tau\geq T} + \gamma(\tau, X_\tau^{0,x})I_{\tau\leq T}$. The related SPDE is

$$\begin{cases} u_t(x) \;=\; h(x) + \displaystyle\int_t^T \left\{ \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u_s(x)}{\partial x_i^2} + f[s, x, u_s(x), \nabla u_s(x)] \right\} ds \\ \qquad\qquad + \displaystyle\int_t^T g[s, x, u_s(x), \nabla u_s(x)]d\overleftarrow{B}_s, \quad x \in \Omega, \; 0 \leq t \leq T \\ u_t(x) \;=\; \gamma(t, x), \quad \text{on } [0, T] \times \partial\Omega \end{cases}$$

3. NUMERICAL ALGORITHMS AND ERROR ESTIMATES

For the simplicity of presentation we only consider the one-dimensional case. The high dimensional cases can be handled through straightforward generalization of the one-dimensional case. To simplify the notations, we shall use (y_t, z_t) to denote the solution $(Y_t^{t,x}, Z_t^{t,x})$ of the BDSDE (4). We also denote $\mathbb{E}_s^{t,x}[X] = \mathbb{E}[X|\mathcal{F}_s^{W,t,x}]$, where $\mathcal{F}_s^{W,t,x} := \sigma(x + W_s - W_t; t \leq s \leq T) \cup \sigma(B_t; 0 \leq t \leq T)$.

3.1 Reference Equations

To further simplify the notations, we denote $f(s, y_s, z_s) = f(s, X_s^{t,x}, y_s, z_s)$ and $g(s, y_s, z_s) = g(s, X_s^{t,x}, y_s, z_s)$, knowing that $x \in \Omega \subset \mathbb{R}$. Then we have

$$y_t = y_{t+\delta} + \int_t^{t+\delta} f(s, y_s, z_s)ds - \int_t^{t+\delta} z_s dW_s + \int_t^{t+\delta} g(s, y_s, z_s)\overleftarrow{dB}_s \tag{8}$$

where δ is a deterministic nonnegative number with $t + \delta \leq T$. Taking the conditional expectation $\mathbb{E}_t^{t,x}[\cdot]$ on Eq. (8), we obtain

$$y_t^{t,x} = \mathbb{E}_t^{t,x}[y_{t+\delta}] + \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s, y_s, z_s)]ds + \int_t^{t+\delta} \mathbb{E}_t^{t,x}[g(s, y_s, z_s)]\overleftarrow{dB}_s \tag{9}$$

where $y_t^{t,x} = \mathbb{E}_t^{t,x}[y_t]$; that is, $y_t^{t,x}$ is the value of y_t at the time-space point (t,x). We use the simple right point formula to approximate the integrals in Eq. (9),

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s,y_s,z_s)]ds = \delta \mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})] + R_y^W \tag{10}$$

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[g(s,y_s,z_s)]\overleftarrow{dB}_s = \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})]\Delta\overleftarrow{B}_t + R_y^B \tag{11}$$

where R_y^W and R_y^B denote the corresponding errors of approximations. Inserting Eqs. (10) and (11) into Eq. (9), we obtain

$$y_t^{t,x} = \mathbb{E}_t^{t,x}[y_{t+\delta}] + \delta\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})] + \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})]\Delta\overleftarrow{B}_t + R_y \tag{12}$$

where $R_y = R_y^W + R_y^B$ is the truncation error for solving y_t. Let $\Delta W_s = W_s - W_t$ for $t \le s \le t+\delta$. Multiplying by $\Delta W_{t+\delta}$ on Eq. (8), taking the conditional expectation $\mathbb{E}_t^{t,x}[\cdot]$, and applying the Itô isometry, we get

$$-\mathbb{E}_t^{t,x}[y_{t+\delta}\Delta W_{t+\delta}] = \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s,y_s,z_s)\Delta W_s]ds + \int_t^{t+\delta} \mathbb{E}_t^{t,x}[g(s,y_s,z_s)\Delta W_s]\overleftarrow{dB}_s - \int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s]ds \tag{13}$$

Similar to Eq. (12), we approximate the integrals in Eq. (13) with the right point formula to obtain

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s,y_s,z_s)\Delta W_s]ds = \delta\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}] + R_{z1}^W \tag{14}$$

$$-\int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s]ds = -\delta z_t^{t,x} + R_{z2}^W \tag{15}$$

and

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[g(s,y_s,z_s)\Delta W_s]\overleftarrow{dB}_s = \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}]\Delta B_t + R_z^B \tag{16}$$

where $z_t^{t,x}$ is the value of z_t at the time-space point (t,x), and R_{z1}^W, R_{z2}^W, and R_z^B are the corresponding approximation errors. Inserting Eqs. (14)–(16) into Eq. (13), we get the second-approximation equation for Eq. (8) as follows:

$$-\mathbb{E}_t^{t,x}[y_{t+\delta}\Delta W_{t+\delta}] = \delta\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}] - \delta z_t^{t,x} + \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}]\Delta B_t + R_z \tag{17}$$

where $R_z = R_{z1}^W + R_{z2}^W + R_z^B$ is the truncation error for solving z_t. Equations (12) and (17) are two key equations of solving BDSDE (8) numerically. We refer to them as reference equations.

3.2 Discrete Scheme

To derive a numerical algorithm from reference equations (12) and (17), we introduce the following time partition on $[0,T]$:

$$\mathcal{R}_{th} = \{t_i | t_i \in [0,T], t_i < t_{i+1}, i = 0, 1, \ldots, N_T - 1, t_0 = 0, t_{N_T} = T\}$$

Let $\Delta t_n = t_{n+1} - t_n$ and $\Delta t = \max_{0 \le n \le N_T - 1} \Delta t_n$. We discretize (12) and (17) by substituting t, δ, y_t, and z_t with t_n, Δt_n with y^n and z^n, respectively, and dropping the errors terms in (12) and (17), to obtain the following numerical algorithm for solving BSDE: given random variable y^N, for $n = N-1, N-2, \ldots, 1, 0$, solve the random variables y^n and z^n backwardly by

$$y^n = \mathbb{E}_{t_n}^{t_n,x}[y^{n+1}] + \Delta t_n \mathbb{E}_{t_n}^{t_n,x} f(t_{n+1}, y^{n+1}, z^{n+1}) + \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y^{n+1}, z^{n+1})]\Delta B_{t_n} \tag{18}$$

and

$$\begin{aligned} 0 \;=\;& \mathbb{E}_{t_n}^{t_n}[y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] \\ +\;& \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}]\Delta B_{t_n} - \Delta t_n z^n \end{aligned} \tag{19}$$

Obviously, (y^n, z^n) is an approximate solution for (y_t, z_t) at $t = t_n, n = 0, 1, \ldots, N_T$.

3.3 Regularity of the Exact Solution

To derive the error estimates, we first need some regularity results for the exact solution. We assume that f and g satisfy the following properties.

$$\begin{aligned} &E\{[f(s,y_1,z_1)-f(t,y_2,z_2)]^2\}\leq L(|s-t|+|y_1-y_2|^2+|z_1-z_2|^2)\\ &E\{[g(s,y_1,z_1)-g(t,y_2,z_2)]^2\}\leq L_1(|s-t|+|y_1-y_2|^2)+L_2|z_1-z_2|^2 \end{aligned} \tag{20}$$

where L, L_1, and L_2 are positive constants and $0\leq L_2<1$ (see [16] for similar assumptions). We also assume that the derivatives f'_x, f'_y, f'_z, g'_x, g'_y, and g'_z of f and g are all continuous and bounded. Let $\nabla y_r^{t,x}$, $\nabla z_r^{t,x}$, and $\nabla X_r^{t,x}$ be the variations of $y_r^{t,x}$, $z_r^{t,x}$, $X_r^{t,x}$ with respect to x at time level $t=r$. Then, the following equation holds:

$$\begin{aligned} \nabla y_s^{t,x} \;=\;& h'(X_T^{t,x})\nabla X_T^{t,x}+\int_s^T[f'_x(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla X_r^{t,x}+f'_y(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla y_r^{t,x}\\ &+\;f'_z(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla z_r^{t,x}]dr+\int_s^T[g'_x(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla X_r^{t,x}+g'_y(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla y_r^{t,x}\\ &+\;g'_z(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla z_r^{t,x}]d\overleftarrow{B}_r-\int_s^T\nabla z_r^{t,x}dW_r \end{aligned} \tag{21}$$

where $\nabla X_r^{t,x}$ is the solution of the following SDE [see [20], p. 464, Eq. (12)].

$$\nabla X_s^{t,x}=1+\int_t^s\partial_x b(r)\nabla X_r^{t,x}dr+\int_t^s\partial_x\sigma(r)\nabla X_r^{t,x}dW_r$$

We have the following result concerning the regularity of the solution (y_t,z_t) of the FBDSDE (4).

Proposition 1.
Assume that Hypothesis (20) holds and the derivatives f'_x, f'_y, f'_z, g'_x, g'_y, and g'_z of f and g are all bounded, then we have

$$E[(y_s^{t,x}-y_t^{t,x})^2]\leq C|s-t| \tag{22}$$

and

$$E[(z_s^{t,x}-z_t^{t,x})^2]\leq C|s-t| \tag{23}$$

Proof: Under the assumptions of the Proposition, pardoux and Peng ([16] proved the estimate (22) and the following estimate

$$\sup_{t\leq s\leq T}E[(y_s^{t,x})^2]\leq C$$

where C is a constant. To obtain (23), we use the fact that (see [16], p. 223, Proposition 2.3)

$$z_s^{t,x}=\nabla y_s^{t,x}(\nabla X_s^{t,x})^{-1}\sigma(X_s^{t,x})$$

and

$$z_t^{t,x}=\nabla y_t^{t,x}\sigma(x)$$

Now, we treat (21) the same as Eq. (4) with f replaced by

$$[f'_x(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla X_r^{t,x}+f'_y(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla y_r^{t,x}+f'_z(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla z_r^{t,x}]$$

and g replaced by

$$[g'_x(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla X_r^{t,x}+g'_y(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla y_r^{t,x}+g'_z(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla z_r^{t,x}]$$

and use the same result of Pardoux and Peng [16] to obtain

$$E[(\nabla y_s^{t,x} - \nabla y_t^{t,x})^2] < C|s-t|$$

and

$$\sup_{t\le s\le T} E[(\nabla y_s^{t,x})^2] \le C$$

Because of the assumption that $b = 0$ and $\sigma = 1$, we have that

$$\nabla X_s^{t,x} = (\nabla X_s^{t,x})^{-1} = 1$$

and $\sigma = 1$. Thus,

$$E[(z_s^{t,x} - z_t^{t,x})^2] = E[(\nabla y_s^{t,x} - \nabla y_t^{t,x})^2] \le C|s-t|$$

3.4 Estimates of Truncation Errors

For the sake of simplicity of our presentation, in the sequel, we use $E_t[\cdot]$ to denote $E_t^{t_n,x}[\cdot]$. Recall the numerical scheme

$$\begin{aligned} y^n &= E_{t_n}[y^{n+1}] + \Delta t_n E_{t_n}[f(t_{n+1}, y^{n+1}, z^{n+1})] + \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y^{n+1}, z^{n+1})] \\ z^n &= \frac{1}{\Delta t_n}\left\{E_{t_n}[y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n E_{t_n}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] \right. \\ &+ \left. \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}]\right\} \end{aligned}$$

and the reference equations

$$\begin{aligned} y_{t_n} &= E_{t_n}[y_{t_{n+1}}] + \Delta t_n E_{t_n}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] + \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y_{t_{n+1}} z_{t_{n+1}})] + R_y^n \\ z_{t_n} &= \frac{1}{\Delta t_n}\left\{E_{t_n}[y_{t_{n+1}}\Delta W_{t_{n+1}}] + \Delta t_n E_{t_n}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{t_{n+1}}] \right. \\ &+ \left. \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{t_{n+1}}]\right\} + \frac{1}{\Delta t_n} R_z^n \end{aligned}$$

where (y_{t_n}, z_{t_n}) is the exact solution. We have truncation errors R_y^n and R_z^n for y and z, respectively, as

$$R_y^n = \int_{t_n}^{t_{n+1}} E_{t_n}[f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]ds + \int_{t_n}^{t_{n+1}} E_{t_n}[g(s, y_s, z_s) - g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]d\overleftarrow{B}_s$$

and

$$\begin{aligned} R_z^n &+ \int_{t_n}^{t_{n+1}} E_{t_n}\{[f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]\Delta W_{t_{n+1}}\}ds \\ &+ \int_{t_n}^{t_{n+1}} E_{t_n}\{[g(s, y_s, z_s) - g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]\Delta W_{t_{n+1}}\}d\overleftarrow{B}_s + \int_{t_n}^{t_{n+1}} E_{t_n}[z_s - z_{t_n}]ds \end{aligned}$$

Denote $f_t = f(t, y_t, z_t)$ and $g_t = g(t, y_t, z_t)$. By Proposition 1, we have the estimates

$$\max_{0\le n\le N-1} E[(y_{t_n} - y_{t_{n-1}})^2] \le C\cdot\Delta t$$

and

$$\max_{0\le n\le N-1} E[(z_{t_n} - z_{t_{n-1}})^2] \le C\cdot\Delta t$$

For the truncation error R_y^n, we have the following estimate.

$$
\begin{aligned}
E[(R_y^n)^2] \;&\leq\; C_1\Delta t_n\int_{t_n}^{t_{n+1}} E\{[f(s,y_s,z_s)-f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]^2\}ds\\
&+\; C_2\int_{t_n}^{t_{n+1}} E\{[g(s,y_s,z_s)-g(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]^2\}ds\\
&\leq\; C_1\Delta t_n\int_{t_n}^{t_{n+1}} E\{[t_{n+1}-s]+(y_s-y_{t_{n+1}})^2+(z_s-z_{t_{n+1}})^2\}ds\\
&+\; C_2\int_{t_n}^{t_{n+1}} E[(t_{n+1}-s)+(y_s-y_{t_{n+1}})^2+(z_s-z_{t_{n+1}})^2]ds \leq C(\Delta t)^2
\end{aligned}
$$

Similarly, we have

$$E[(R_z^n)^2]\leq K(\Delta t)^3$$

3.5 Error Estimate for y

Denote $e_y^n = y_{t_n}-y^n$, $e_z^n = z_{t_n}-z^n$, $e_f^n = f(t_n,y_{t_n},z_{t_n})-f(t_n,y^n,z^n)$, and $e_g^n = g(t_n,y_{t_n},z_{t_n})-g(t_n,y^n,z^n)$. We have the estimate of e_y^n for scheme (18) and (19) in the following theorem.

Theorem 1.
Let (y_t,z_t) be the exact solution and (y^n,z^n) be the solution of the scheme (18) and (19). If Hypothesis (20) is satisfied and the derivatives f'_x, f'_y, f'_z, g'_x, g'_y, and g'_z of f and g are all bounded, then

$$\max_{0\leq n\leq N-1} E[e_y^n]^2\leq C\Delta t$$

where C is a constant.

Proof: We first decompose the error for y as

$$e_y^n = E_{t_n}[e_y^{n+1}]+E_{t_n}[e_f^{n+1}]\Delta t_n + E_{t_n}[e_g^{n+1}]\Delta B_{t_{n+1}}+R_y^n$$

Taking square on both sides of the above equation and then taking expectation, we obtain

$$
\begin{aligned}
E|e_y^n|^2 \;&=\; E\big\{(E_{t_n}[e_y^{n+1}]+\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]+R_y^n)^2+(\Delta t_nE_{t_n}[e_f^{n+1}])^2\\
&+\; 2(E_{t_n}[e_y^{n+1}]+\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]+R_y^n)(\Delta t_nE_{t_n}[e_f^{n+1}])\big\}\\
&=\; E\left\{|E_{t_n}[e_y^{n+1}]|^2+(\Delta B_{t_{n+1}})^2|E_{t_n}[e_g^{n+1}]|^2+(R_y^n)^2+2E_{t_n}[e_y^{n+1}]\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]\right.\\
&+\; \left.2E_{t_n}[e_y^{n+1}]R_y^n+2\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]R_y^n\right\}+E\{(\Delta t_nE_{t_n}[e_f^{n+1}])^2\}\\
&+\; E\{2(E_{t_n}[e_y^{n+1}]+\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]+R_y^n)(\Delta t_nE_{t_n}[e_f^{n+1}])\} = A_y+B_y+C_y
\end{aligned}
\tag{24}
$$

where

$$
\begin{aligned}
A_y \;&=\; E\left\{|E_{t_n}[e_y^{n+1}]|^2+(\Delta B_{t_{n+1}})^2|E_{t_n}[e_g^{n+1}]|^2+(R_y^n)^2+2E_{t_n}[e_y^{n+1}]\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]\right.\\
&+\; \left.2E_{t_n}[e_y^{n+1}]R_y^n+2\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]R_y^n\right\}
\end{aligned}
$$

$$B_y = E\{(\Delta t_nE_{t_n}[e_f^{n+1}])^2\}$$

and

$$C_y = E\{2(E_{t_n}[e_y^{n+1}]+\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]+R_y^n)(\Delta t_nE_{t_n}[e_f^{n+1}])\}$$

Next, we use Cauchy's inequality and Young's inequality and the facts that

$$E\left\{E_{t_n}[e_y^{n+1}]\cdot\int_{t_n}^{t_{n+1}} E_{t_n}[g(s,y_s,z_s)-g(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]d\overleftarrow{B}_s\right\}=0$$

and

$$\int_{t_n}^{t_{n+1}} E[f(s,y_s,z_s)-f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]^2 ds \le C(\Delta t)^2$$

to obtain

$$\begin{aligned}
A_y &= E\left\{|E_{t_n}[e_y^{n+1}]|^2 + (\Delta B_{t_{n+1}})^2 |E_{t_n}[e_g^{n+1}]|^2 + (R_y^n)^2 + 2E_{t_n}[e_y^{n+1}]\Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}]\right.\\
&+ \left. 2E_{t_n}[e_y^{n+1}]R_y^n + 2\Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}]R_y^n\right\} \le E\{|E_{t_n}[e_y^{n+1}]|^2\} + \Delta t_n E\{|E_{t_n}[e_g^{n+1}]|^2\}\\
&+ E[(R_y^n)^2] + 2E\left\{E_{t_n}[e_y^{n+1}]\cdot \int_{t_n}^{t_{n+1}} E_{t_n}^x[f(s,y_s,z_s)-f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]ds + E_{t_n}[e_y^{n+1}]\right.\\
&\cdot \left.\int_{t_n}^{t_{n+1}} E_{t_n}^x[g(s,y_s,z_s)-g(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]d\overleftarrow{B}_s\right\} + \Delta t_n \epsilon_1 E\{|E_{t_n}[e_g^{n+1}]|^2\} + E\left[\frac{1}{\epsilon_1}(R_y^n)^2\right]\\
&\le E\{|E_{t_n}[e_y^{n+1}]|^2\} + \Delta t_n E\{|E_{t_n}[e_g^{n+1}]|^2\} + E[(R_y^n)^2] + \Delta t_n E[e_y^{n+1}]^2\\
&+ \int_{t_n}^{t_{n+1}} E[f(s,y_s,z_s)-f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]^2 ds + \Delta t_n \epsilon_1 E\{|E_{t_n}[e_g^{n+1}]|^2\} + E\left[\frac{1}{\epsilon_1}(R_y^n)^2\right]\\
&\le E\{|E_{t_n}[e_y^{n+1}]|^2\} + \Delta t_n E\{(1+\epsilon_1)|E_{t_n}[e_g^{n+1}]|^2\} + \Delta t_n E[e_y^{n+1}]^2 + C_1(\Delta t)^2
\end{aligned} \tag{25}$$

where $\epsilon_1 > 0$ is a constant to be determined later. By the Lipschitz continuity of f, we have

$$B_y = E\{(\Delta t_n E_{t_n}[e_f^{n+1}])^2\} \le L(\Delta t_n)^2 (E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2) \tag{26}$$

Similarly for C_y, using Cauchy's inequality and Young's inequality, we obtain

$$\begin{aligned}
C_y &= E\{2(E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + R_y^n)(\Delta t_n E_{t_n}[e_f^{n+1}])\}\\
&\le \Delta t_n \frac{1}{\epsilon_2} E\{E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + R_y^n\}^2 + \Delta t_n \epsilon_2 E[e_f^{n+1}]^2\\
&\le \Delta t_n \frac{3}{\epsilon_2} E\{E_{t_n}[e_y^{n+1}]^2 + \Delta t_n (L_1 E_{t_n}[e_y^{n+1}]^2 + L_2 E_{t_n}[e_z^{n+1}]^2) + (R_y^n)^2\}\\
&+ \Delta t_n E\{L\epsilon_2 E_{t_n}[e_y^{n+1}]^2 + L\epsilon_2 E_{t_n}[e_z^{n+1}]^2\} \le \Delta t_n C_2 E[e_y^{n+1}]^2\\
&+ (\Delta t_n)^2 C_3 (E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2) + \Delta t_n L\epsilon_2 E[e_z^{n+1}]^2 + C_4(\Delta t)^2
\end{aligned} \tag{27}$$

where $\epsilon_2 > 0$ is a constant to be determined later. Combining (24)–(27) together, we obtain

$$\begin{aligned}
E[e_y^n]^2 &\le E\{|E_{t_n}[e_y^{n+1}]|^2\} + \Delta t_n E\{(1+\epsilon_1)|E_{t_n}[e_g^{n+1}]|^2\} + \Delta t_n L\epsilon_2 E[e_z^{n+1}]^2 + K_1 \Delta t_n E[e_y^{n+1}]^2\\
&+ K_2(\Delta t_n)^2 E[e_z^{n+1}]^2 + K_3(\Delta t)^2
\end{aligned} \tag{28}$$

For e_z^n, we have the identity

$$\Delta t_n e_z^n = E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n \tag{29}$$

Taking the square on both sides of Eq. (29) and then taking expectation, we obtain

$$\begin{aligned}
E[\Delta t_n e_z^n]^2 &= E\{E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n\}^2 + E\{(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2\}\\
&+ 2E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n)(\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])\}\\
&= E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2 (E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2\\
&+ 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])\Delta B_{t_{n+1}} + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])R_z^n\\
&+ 2E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}} R_z^n\} + E\{(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2\}\\
&+ 2E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n)(\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])\}\\
&= A_z + B_z + C_z
\end{aligned} \tag{30}$$

where

$$A_z = E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2 + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}]) \times (E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])\Delta B_{t_{n+1}} + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])R_z^n + 2E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}} R_z^n\}$$

$$B_z = E\{(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2\}$$

and

$$C_z = 2E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n \cdot (\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])\}$$

For any $\mathcal{F}_t^W$ adapted process X_t, we have

$$\begin{aligned}(E_{t_n}[X_{t_{n+1}}\Delta W_{t_{n+1}}])^2 &= \{E_{t_n}[(X_{t_{n+1}} - E_{t_n}[X_{t_{n+1}}])\Delta W_{t_{n+1}}]\}^2 \le E_{t_n}\{(X_{t_{n+1}} - E_{t_n}[X_{t_{n+1}}])\}^2 \Delta t_n \\ &= \Delta t_n\{E_{t_n}[(X_{t_{n+1}})^2] - |E_{t_n}[X_{t_{n+1}}]|^2\}\end{aligned} \tag{31}$$

For A_z, using Eq. (31), Cauchy's inequality and Young's inequality, we have

$$\begin{aligned}A_z &= E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2 + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}]) \\ &\times (E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])\Delta B_{t_{n+1}} + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])R_z^n + 2E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}} R_z^n\} \\ &\le E\Bigg\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2 + \epsilon(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 \\ &+ \frac{1}{\epsilon}(R_z^n)^2 + \epsilon_1(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}})^2 + \frac{1}{\epsilon_1}(R_z^n)^2\Bigg\} \le \Delta t_n E[E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2] \\ &+ (\Delta t_n)^2 E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2] + E[(R_z^n)^2] + \epsilon\Delta t_n E\{E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2\} \\ &+ \frac{1}{\epsilon}E[(R_z^n)^2] + \epsilon_1(\Delta t_n)^2 E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2] + \frac{1}{\epsilon_1}E[(R_z^n)^2] \\ &\le (1+\epsilon)\Delta t_n E\{E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2\} + (1+\epsilon_1)(\Delta t_n)^2 E\{E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2\} \\ &+ C_5(\Delta t)^3\end{aligned} \tag{32}$$

Under the conditions in the theorem, we have

$$B_z = E[(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2] \le C_6(\Delta t)^3 \tag{33}$$

Similarly, using Cauchy's inequality and Young's inequality, we obtain

$$\begin{aligned}C_z &= 2E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n)(\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])\} \\ &\le \Delta t_n \frac{1}{\epsilon_2} E\{(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n)\}^2 + \Delta t_n \epsilon_2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2 \\ &\le (\Delta t_n)^2 \frac{3}{\epsilon_2}\{E[e_y^{n+1}]^2 + \Delta t_n L_1 E[e_y^{n+1}]^2 + \Delta t_n L_2 E[e_z^{n+1}]^2 + (R_z^n)^2\} \\ &+ (\Delta t_n)^2 L\epsilon_2\{E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2\} \le C_7\left\{(\Delta t_n)^2 E[e_y^{n+1}]^2 + (\Delta t_n)^3(E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2)\right\} \\ &+ (\Delta t_n)^2 L\epsilon_2 E[e_z^{n+1}]^2 + C_8(\Delta t)^3\end{aligned} \tag{34}$$

Here, ϵ_1 and ϵ_2 are same as in Eq. (28) and ϵ is a positive constant, which will be determined later. Combining (30) and (32)–(34) together, we get

$$\begin{aligned}E[\Delta t_n e_z^n]^2 &\le (1+\epsilon)\Delta t_n E\{E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2\} + (1+\epsilon_1)(\Delta t_n)^2 E\{E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2\} \\ &+ (\Delta t_n)^2 L\epsilon_2 E[e_z^{n+1}]^2 + K_4(\Delta t_n)^2 E_{t_n}[e_y^{n+1}]^2 + K_5(\Delta t_n)^3 E_{t_n}[e_z^{n+1}]^2 + K_6(\Delta t)^3\end{aligned} \tag{35}$$

Next we divide by $\Delta t_n(1+\epsilon)$ on both sides of Eq. (35) to obtain

$$\begin{aligned} \frac{\Delta t_n}{1+\epsilon}E[e_z^n]^2 \;\le\; & E\{E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2\} + (\Delta t_n)\frac{1+\epsilon_1}{1+\epsilon}E\{E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2\} \\ &+ (\Delta t_n)L\frac{\epsilon_2}{1+\epsilon}E[e_z^{n+1}]^2 + K_4\Delta t_n E_{t_n}[e_y^{n+1}]^2 + K_5(\Delta t_n)^2 E_{t_n}[e_z^{n+1}]^2 + K_6(\Delta t)^2 \end{aligned} \tag{36}$$

Adding (36) to (28), we obtain

$$\begin{aligned} E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon}E[e_z^n]^2 \;\le\; & E[|E_{t_n}[e_y^{n+1}]|^2] + \Delta t_n E[(1+\epsilon_1)|E_{t_n}[e_g^{n+1}]|^2] + \Delta t_n L\epsilon_2 E[e_z^{n+1}]^2 \\ &+ K_1\Delta t_n E[e_y^{n+1}]^2 + K_2(\Delta t_n)^2 E[e_z^{n+1}]^2 + K_3\left\{h^2 + E\left[\int_{t_n}^{t_{n+1}}|z_s - z_{t_n}|^2\right]ds\right\} \\ &+ E\{E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2\} + (\Delta t_n)\frac{1+\epsilon_1}{1+\epsilon}E\{E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2\} \\ &+ (\Delta t_n)L\frac{\epsilon_2}{1+\epsilon}E[e_z^{n+1}]^2 + K_4\Delta t_n E_{t_n}[e_y^{n+1}]^2 + K_5(\Delta t_n)^2 E_{t_n}[e_z^{n+1}]^2 \\ &+ K_6\left\{(\Delta t)^2 + E\left[\int_{t_n}^{t_{n+1}}|z_s - z_{t_n}|^2\right]ds\right\} = E[e_y^n]^2 + (\Delta t_n)\frac{1+\epsilon_1}{1+\epsilon}E\{E_{t_n}[e_g^{n+1}]^2\} \\ &+ (\Delta t_n)\frac{1+\epsilon_1}{1+\epsilon}\epsilon E\{|E_{t_n}[e_g^{n+1}]|^2\} + (\Delta t_n)L\frac{\epsilon_2+\epsilon_2+\epsilon\epsilon_2}{1+\epsilon}E[e_z^{n+1}]^2 \\ &+ G_1\Delta t_n E[e_y^{n+1}]^2 + G_2(\Delta t_n)^2 E[e_z^{n+1}]^2 + G_3(\Delta t)^2 \\ \le\; & E[e_y^n]^2 + (\Delta t_n)\left(L_2\frac{1+\epsilon_1}{1+\epsilon} + L_2\frac{1+\epsilon_1}{1+\epsilon}\epsilon + L\frac{\epsilon_2+\epsilon_2+\epsilon\epsilon_2}{1+\epsilon}\right)E[e_z^{n+1}]^2 \\ &+ G_1\Delta t_n E[e_y^{n+1}]^2 + G_2(\Delta t_n)^2 E[e_z^{n+1}]^2 + G_3(\Delta t)^2 \end{aligned} \tag{37}$$

Now we choose ϵ, ϵ_1, and ϵ_2, all positive, sufficiently small such that

$$L_2(1+\epsilon_1) + L_2\epsilon(1+\epsilon_1) + L(2\epsilon_2 + \epsilon\epsilon_2) \le 1$$

This is possible because $L_2 < 1$. Thus, by Eq. (37), we have

$$E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon}E[e_z^n]^2 \le E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon}E[e_z^{n+1}]^2 + T_1\Delta t_n\left\{E[e_y^{n+1}]^2 + \frac{\Delta t_n}{1+\epsilon}E[e_z^{n+1}]^2\right\} + T_2(\Delta t)^2$$

Denote $e_n := E[e_y^n]^2 + [\Delta t_n/(1+\epsilon)]E[e_z^n]^2$. Then, the above equation becomes

$$e_n \le (1 + T_1\Delta t)e_{n+1} + T_2(\Delta t)^2$$

By Gronwall's inequality, we have

$$\max_{0\le n\le N-1}\left\{E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon}E[e_z^n]^2\right\} \le C\Delta t$$

as required.

3.6 Error Estimate for z

We first construct an approximate solution $(\tilde{Y}, \tilde{Z})$ with step process as follows. Let

$$\tilde{Y}_{t_{n+1}} = y^{n+1} + \Delta t_n \cdot f^{n+1} + \Delta B_{t_n} \cdot g^{n+1}$$

and $\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B$. By an extension of Itô's martingale representation theorem, we can find an $\mathcal{G}_t$ adapted process $\tilde{Z}_t$, such that

$$\tilde{Y}_{t_{n+1}} = E[\tilde{Y}_{t_{n+1}}|\mathcal{G}_{t_n}] + \int_{t_n}^{t_{n+1}} \tilde{Z}_r dW_r \tag{38}$$

Define a continuous approximate process $(\tilde{Y}, \tilde{Z})$ as follows:

$$\tilde{Y}_t = y^{n+1} + f^{n+1} \cdot (t_{n+1} - t) + g^{n+1} \cdot (B_{t_{n+1}} - B_t) - \int_t^{t_{n+1}} \tilde{Z}_r dW_r,\ t \in (t_n, t_{n+1}],\ n = 0, \cdots, N-1 \tag{39}$$

where

$$f^{n+1} = f(t_{n+1}, y^{n+1}, z^{n+1})$$

and

$$g^{n+1} = g(t_{n+1}, y^{n+1}, z^{n+1})$$

By Eq. (19) and (38) it is easy to see that

$$\Delta t_n z^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}[\tilde{Z}_r] dr$$

Thus,

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} E[(z_s - z^n)^2] ds = \int_{t_n}^{t_{n+1}} E\left[\left(z_s - \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}[\tilde{Z}_r] dr\right)^2\right] ds \\
&= \int_{t_n}^{t_{n+1}} E\left[\left(\frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}[z_s - \tilde{Z}_r] dr\right)^2\right] ds \le E\int_{t_n}^{t_{n+1}} \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}[(z_s - \tilde{Z}_r)^2] dr ds \\
&\le 2\left(\int_{t_n}^{t_{n+1}} E\{\mathbb{E}_{t_n}[(z_r - \tilde{Z}_r)^2]\} dr + \int_{t_n}^{t_{n+1}} \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} E\{\mathbb{E}_{t_n}[(z_s - z_r)^2]\} dr ds\right) \\
&\le 2\left\{\int_{t_n}^{t_{n+1}} E[(z_r - \tilde{Z}_r)^2] dr + (\Delta t_n)^2\right\}
\end{aligned} \tag{40}$$

Now, we are ready to prove an error estimate for z.

Theorem 2.
Let (y_t, z_t) be the exact solution and (y^n, z^n) be the solution of the scheme (18) and (19). Assume that hypothesis (20) holds and derivatives f'_x, f'_y, f'_z, g'_x, g'_y, and g'_z of f and g are all bounded. Then for Δt sufficiently small, we have

$$\sum_{n=0}^{N-1} E\int_{t_n}^{t_{n+1}} (z_s - z^n)^2 ds \le C\Delta t$$

Proof: For $t \in [t_n, t_{n+1}]$, let $e_y^t = y_t - \tilde{Y}_t$, $e_z^t = z_t - \tilde{Z}_t$, $f_t = f(t, y_t, z_t)$ and $g_t = g(t, y_t, z_t)$. Subtracting BDSDE (39) from Eq. (4), we have that

$$e_y^t = e_y^{t_{n+1}} + \int_t^{t_{n+1}} (f_s - f^{n+1}) ds + \int_t^{t_{n+1}} (g_s - g^{n+1}) d\overleftarrow{B}_s - \int_t^{t_{n+1}} e_z^s dW_s \tag{41}$$

Taking the square on both sides of Eq. (41), applying Itô's formula [16] and taking expectation, we have

$$
\begin{aligned}
&E[(e_y^t)^2] + E\int_t^{t_{n+1}} (e_z^s)^2 ds = E[(e_y^{t_{n+1}})^2] + 2E\int_t^{t_{n+1}} e_y^s \cdot (f_s - f^{n+1}) ds + E\int_t^{t_{n+1}} (g_s - g^{n+1})^2 ds \\
&\le E[(e_y^{t_{n+1}})^2] + \frac{1}{\epsilon_0} E\int_t^{t_{n+1}} (e_y^s)^2 ds + \epsilon_0 E\int_t^{t_{n+1}} (f_s - f^{n+1})^2 ds + E\int_t^{t_{n+1}} (g_s - g^{n+1})^2 ds \\
&\le E[(e_y^{t_{n+1}})^2] + \frac{1}{\epsilon_0} E\int_t^{t_{n+1}} (e_y^s)^2 ds + E\int_t^{t_{n+1}} 2c(y_s - y^{n+1})^2 + (\epsilon_1 + \alpha)(z_s - z^{n+1})^2 ds \\
&\le E[(e_y^{t_{n+1}})^2] + \frac{1}{\epsilon_0} E\int_t^{t_{n+1}} (e_y^s)^2 ds + CE\int_t^{t_{n+1}} (y_s - y_{t_{n+1}})^2 + (y_{t_{n+1}} - y^{n+1})^2 ds \\
&+E\int_t^{t_{n+1}} \left(\epsilon_1 + \frac{1}{\epsilon_2} + \alpha\right) \left(\left\{ z_s - (\Delta t_n)^{-1} \int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r] dr \right\}^2 \right) \\
&+(\epsilon_1 + \epsilon_2 + \alpha) \left(\left\{ (\Delta t_n)^{-1} \int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r] dr - z^{n+1} \right\}^2 \right) ds
\end{aligned} \tag{42}
$$

where ϵ_0, ϵ_1, and ϵ_2 are positive constants to be determined later. Because $E[(y_s - y_t)^2] + E[(z_s - z_t)^2] \le C|s-t|$, we have

$$
E\int_t^{t_{n+1}} (y_s - y_{t_{n+1}})^2 ds \le C(\Delta t)^2
$$

and

$$
\begin{aligned}
&E\int_t^{t_{n+1}} \left\{ z_s - (\Delta t_n)^{-1} \int_{t_n+1}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r] dr \right\}^2 ds = E\int_t^{t_{n+1}} \left\{ z_s - z_{t_{n+1}} + z_{t_{n+1}} - (\Delta t_n)^{-1} \int_{t_n+1}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r] dr \right\}^2 ds \\
&= 2E\int_t^{t_{n+1}} (z_s - z_{t_{n+1}})^2 + \left\{ (\Delta t_n)^{-1} \int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_{t_{n+1}} - z_r] dr \right\}^2 ds \le C(\Delta t)^2
\end{aligned}
$$

Also, because $\Delta t z^{n+1} = \int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[\tilde{Z}_r] dr$, we have

$$
\left\{ (\Delta t_n)^{-1} \int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r] dr - z^{n+1} \right\}^2 \le (\Delta t_n)^{-1} \int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[(z_r - \tilde{Z}_r)^2] dr
$$

Thus, we can rewrite (42) as

$$
\begin{aligned}
&E[(e_y^t)^2] + E\int_t^{t_{n+1}} (e_z^s)^2 ds \le E[(e_y^{t_{n+1}})^2] + CE\int_t^{t_{n+1}} (e_y^{t_{n+1}}) ds + (\epsilon_1 + \epsilon_2 + \alpha) E\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2 \\
&+\frac{1}{\epsilon_0} E\int_t^{t_{n+1}} (e_y^s)^2 ds
\end{aligned}
$$

Choose ϵ_1 and ϵ_2 small enough such that $(\epsilon_1 + \epsilon_2 + \alpha) = K < 1$ (because $\alpha < 1$). Then,

$$
\begin{aligned}
&E[(e_y^t)^2] + E\int_t^{t_{n+1}} (e_z^s)^2 ds \le C_1 E\int_t^{t_{n+1}} (e_y^s)^2 ds + E[(e_y^{t_{n+1}})^2] + C_2 E\int_{t_n}^{t_{n+1}} (e_y^{t_{n+1}})^2 ds \\
&+KE\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2 \le \left\{ (1 + C_2 \Delta t) E[(e_y^{t_{n+1}})^2] + KE\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2 \right\} \\
&+C_1 E\int_t^{t_{n+1}} (e_y^s)^2 ds
\end{aligned} \tag{43}
$$

By Gronwall's inequality, we get

$$E[(e_y^s)^2] \leq C\left\{(1+C_2\Delta t)E[(e_y^{t_{n+1}})^2] + E\int_{t_{n+1}}^{t_{n+2}}(e_z^s)^2 ds + C(\Delta t)^2\right\} \tag{44}$$

for $s \in [t_n, t_{n+1}]$. Now, we let $t = t_n$ in Eq. (43) and substitute (44) in (43) to obtain

$$E[(e_y^{t_n})^2] + E\int_{t_n}^{t_{n+1}}(e_z^s)^2 ds \leq (1+C_1\Delta t)E[(e_y^{t_{n+1}})^2] + (K + C_2\Delta t)E\int_{t_{n+1}}^{t_{n+2}}(e_z^s)^2 ds + C(\Delta t)^2$$

Now, by Theorem 1, we easily obtain

$$E[(e_y^{t_n})^2] + E\int_{t_n}^{t_{n+1}}(e_z^s)^2 ds \leq E[(e_y^{t_{n+1}})^2] + (K + C\Delta t)E\int_{t_{n+1}}^{t_{n+2}}(e_z^s)^2 ds + C(\Delta t)^2$$

Let Δt be sufficiently small such that $C\Delta t + K \leq L < 1$, where L is a constant. Summing the above equation from $n = 0$ to $n = N-1$, we obtain

$$(1-L)\sum_{n=0}^{N-1} E\int_{t_n}^{t_{n+1}}(e_z^s)^2 ds \leq C\Delta t + L\int_{t_{N-1}}^{t_N} E[(e_z^s)^2] ds \leq C\Delta t \tag{45}$$

Through a similar argument, it is easy to obtain

$$\int_{t_{N-1}}^{t_N} E[(e_z^s)^2] ds \leq C\Delta t \tag{46}$$

By Eqs. (40), (45) and (46), we conclude that

$$\sum_{n=0}^{N-1} E\int_{t_n}^{t_{n+1}}(z_s - z^n)^2 ds \leq C\Delta t$$

as required in Theorem 2.

4. NUMERICAL EXPERIMENTS

In this section, we carry out numerical experiments to verify the rate of convergence results obtained in Section 3 and compare our numerical method to the finite difference method for stochastic parabolic partial differential equations [6]. The conditional expectations in Eqs. (18) and (19) can be evaluated using Monte Carlo method or Gaussian quadratures [21, 22]. In our examples, we use the binomial tree method which is amount to two point Gaussian quadrature [5].

Example 1: In the first example, we consider the initial boundary value problem,

$$\begin{aligned}
u_t(x) &= \exp(x\cdot T)\sin\left[\frac{B(T)}{2}\right] + \int_t^T \left[\frac{1}{2}\frac{\partial^2}{\partial x^2}u_s(x) - \left(x + \frac{1}{8}\right)u_s(x) - \frac{s}{2}\frac{\partial}{\partial x}u_s(x)\right] ds \\
&+ \int_t^T \frac{1}{2}\exp(x\cdot s)\cos\left[B(T) - \frac{B(s)}{2}\right] d\overleftarrow{B}_s \\
u_t(-1) &= \exp[(-1)\cdot t]\sin\left[B(T) - \frac{B(T)}{2}\right] \\
u_t(1) &= \exp(1\cdot t)\sin\left[B(T) - \frac{B(t)}{2}\right]
\end{aligned} \tag{47}$$

We construct the SPDE (47) in such a way that $u_t(x) = \exp(x \cdot t)\sin\{B(T) - [B(t)/2]\}$ is the exact solution. The corresponding BDSDE is given by

$$y_0^{0,x} = \exp(X_T^{0,x} \cdot T)\sin\left[\frac{B(T)}{2}\right] I_{\tau \geq T} + \exp(X_\tau^{0,x} \cdot \tau)\sin\left[B(T) - \frac{B(\tau)}{2}\right] I_{\tau \leq T}$$

$$+ \int_0^{T\wedge\tau}\left[-\left(X_t^{0,x} + \frac{1}{8}\right)y_t^{0,x} - \frac{t}{2}z_t^{0,x}\right]ds - \int_0^{T\wedge\tau} z_t^{0,x}\,dW_t + \int_0^{T\wedge\tau}\frac{1}{2}\exp(X_t^{0,x}\cdot t)\cos\left[B(T) - \frac{B(t)}{2}\right]d\overleftarrow{B}_t$$

The numerical results are shown in Table 1 and Fig. 1. Here, J denotes the number of spatial partition grids, N_T (FD) the number of time steps used in finite difference method, N_T (BDSDE) the number of time steps used in our method for solving the related BDSDE, and error (FD) and error (BDSDE) the errors of finite difference method and our method, respectively. The results indicate that our algorithm is comparable to the the algorithm of solving the SPDE directly using the finite difference scheme, with a little higher rate of convergence.

Example 2: In this example, we consider the unbounded SPDE initial value problem.

$$\begin{aligned} u_t(x) &= \sin(x+T)\cos(2B_T) + \int_t^T \left[\frac{1}{2}\frac{\partial^2}{\partial x^2}u_s(x) - \frac{\partial}{\partial x}u_s(x)\right]ds \\ &+ \int_t^T \sin[W(s)+s][\sin(B_T+B_s) - \cos(B_T+B_s)] + u_s(x)\,d\overleftarrow{B}_s \end{aligned} \tag{48}$$

TABLE 1: Example 1

J	N_T (FD)	Error (FD)	N_T (BDSDE)	Error (BDSDE)
2^2	2^5	0.0213	2^4	0.200
2^3	2^7	0.0177	2^6	0.0383
2^4	2^9	0.0112	2^8	0.0106
2^5	2^{11}	0.00587	2^{10}	0.00376
2^6	2^{13}	0.00313	2^{12}	0.00152

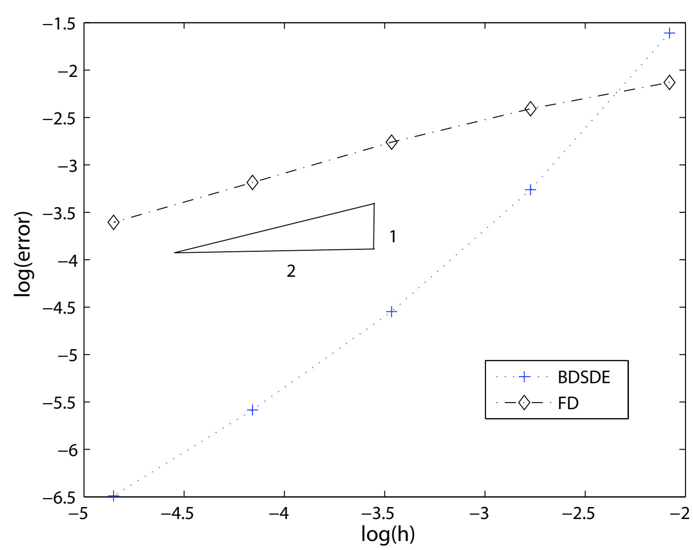

FIG. 1: Example 1: Convergence comparison between the direct finite difference scheme and our scheme.

where $u_t(x) = \sin(x+t)\cos(B_T + B_t)$ is the solution of the SPDE (48). The corresponding FBDSDE is given by

$$y_0^{0,x} = \sin[W(T)+T]\cos(2B_T) - \int_0^T z_s^{0,x} ds + \int_0^t \{\sin[W(s)+s][\sin(B_T+B_s) - \cos(B_T+B_s)] +_s^{0,x}\} d\overleftarrow{B}_s - \int_0^t z_s^{0,x} dW_s$$

The errors are shown in Table 2 and Fig. 2, in which error (Y) and error (Z) are errors for Y and Z at time-space point $(t,x) = (0,0)$, respectively. These data also confirm our rate of converence results.

5. CONCLUSION

We constructed a numerical algorithm for FBDSDEs based on our reference equation formulation. Rigorous numerical analysis proves the half order rate of convergence for this algorithm. Through an equivalence relation between FBDSDEs and a class of stochastic PDEs (Zakai equations), our algorithm can also be used to solve these SPDEs numerically. The rate of convergence of our algorithm is comparable to the general finite difference algorithms for stochastic parabolic PDEs. Through stochastic Taylor expansion, our methodology has the potential of deriving even higher order algorithms. Such a research is underway, and the results will be reported elsewhere.

TABLE 2: Example 2

J	N_T	Error (Y) (u)	Error (Z) (∇u)
2^3	2^3	$6.4096E-002$	0.1188
2^4	2^4	$3.2019E-002$	$9.0028E-002$
2^5	2^5	$1.4426E-002$	$6.3314E-002$
2^6	2^6	$7.2577E-003$	$4.1337E-002$
2^7	2^7	$3.5995E-003$	$2.7149E-002$

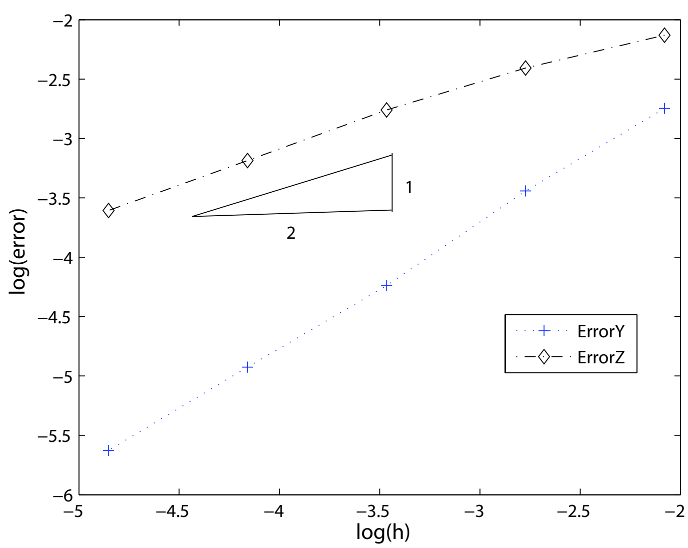

FIG. 2: Example 2: Convergence comparison between the approximations of y and z.

ACKNOWLEDGMENTS

This research is partially supported by Air Force Office of Scientific Research under Grant No. FA9550-08-1-0119, National Science Foundation under Grant No. DMS0914554, and by Guangdong Provincial Government of China through the “Computational ScienceInnovative Research Team” program.

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