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A NEW GIBBS SAMPLING BASED BAYESIAN MODEL UPDATING APPROACH USING MODAL DATA FROM MULTIPLE SETUPS

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This paper presents a new Gibbs sampling based approach for Bayesian model updating of a linear dynamic system based on modal data (natural frequencies and partial mode shapes of some of the dominant modes) obtained from a structure using multiple setups. Modal data from multiple setups pose a problem as mode shapes identified from multiple setups are normalized individually and the scaling factors to form the overall mode shape are not known a priori. For comprehensive quantification of the uncertainties, the proposed approach allows for an efficient update of the probability distribution of the model parameters, overall mode shapes, scaling factors, and prediction error variances. The proposed approach does not require solving the eigenvalue problem of any structural model or matching of model and experimental modes, and is robust to the dimension of the problem. The effectiveness and efficiency of the proposed method are illustrated by simulated numerical examples.

KEY WORDS: *Bayesian model updating, Gibbs sampling, multiple setups, stochastic simulation, uncertainty quantification*

1. INTRODUCTION

Dynamic characteristics of a vibrating structure are usually predicted by finite element (FE) models. These models often give results that differ from the experimentally obtained data because of the uncertainties involved in the modeling process. These uncertainties may arise because of several factors such as incomplete knowledge or simplifying assumptions made during the modeling process, uncertainties about modeling parameters, incompleteness of experimentally obtained data or measurement errors. Accurate and appropriate uncertainty characterization of the system modeling parameters and the modeling errors is essential for a robust prediction of future response and reliability of structures [1–4]. Therefore, FE models need to be updated using the experimentally obtained data such that they more accurately reflect the dynamic behavior of the structure of interest.

In recent years, much focus has been on probabilistic model updating [5–10] for comprehensive quantification of the uncertainties. Beck and Katafygiotis [5] presented a general Bayesian statistical framework for model updating that explicitly treats model uncertainties and prediction errors. For a given model $\mathcal{M}$, given the prior probability distribution function (PDF) $p(\theta|\mathcal{M})$ of the uncertain model parameters $\theta \in \Theta \in \mathbb{R}^{n_\theta}$ and data D, the posterior PDF $p(\theta|D, \mathcal{M})$ is given by

$$p(\theta|D,\mathcal{M}) = \frac{p(D|\theta,\mathcal{M})p(\theta|\mathcal{M})}{p(D|\mathcal{M})} \tag{1}$$

where $p(D|\theta, \mathcal{M})$ is the likelihood function and $p(D|\mathcal{M})$ is the normalizing constant such that the integral of the right-hand side of Eq. (1) over the domain of θ is equal to unity. The denominator in Eq. (1) is often not known explicitly, thus $p(\theta|D, \mathcal{M})$ is only known up to a normalizing constant. Based on the topology of $p(\theta|D, \mathcal{M})$, Katafygiotis and Beck [11] proposed to classify the model class $\mathcal{M}$ into three different categories: globally identifiable (unique

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optimal estimate), locally identifiable (finite number of optimal estimates), and unidentifiable (continuum of optimal estimates). Beck and Katafygiotis [5] adopted Laplace's method of asymptotic approximation which requires a non-convex optimization [12] to obtain the posterior PDF of the model parameters. However, the accuracy of such an approximation is questionable when either the amount of data is not sufficiently large or the chosen class of models turn out to be unidentifiable based on the available data. Also, the approach is computationally challenging especially in a high-dimensional parameter space or when the model class is not globally identifiable. To avoid these limitations, lately focus has shifted to stochastic simulation methods for Bayesian updating especially Markov Chain Monte Carlo (MCMC) methods, such as Gibbs sampling [8], Metropolis–Hasting [13], Transition MCMC [14], and hybrid Monte Carlo [15] algorithms. Stochastic simulation methods allow generating samples which are distributed as posterior PDF without the need of evaluating the normalizing constant in Bayes' theorem.

In structural dynamics, the experimentally obtained data can consist of data in the frequency or time domain, or modal data. The proposed approach makes use of the modal data to update the model parameters of a linear dynamic system. It is assumed that the modal data are obtained using low-amplitude vibrations so the assumption that the structure behave approximately linearly during such vibration is valid. There are several ambient or forced vibration based techniques available in the literature for modal identification [16–19].

In the past, several authors have considered the problem of model updating using modal data. Vanik el al. [4] used non-linear optimization to calculate the most probable values of the model parameters [that is, θ that maximizes $p(\theta|D, \mathcal{M})$, for the globally identifiable case] using modal data. Yuen et al. [20] calculated the most probable values of the model parameters using modal data by iteratively solving a series of coupled linear optimization problems. The most probable model does not imply that this model can exactly represent the dynamic behavior of structure of interest. Instead, there is a region in the uncertain parameter space corresponding to high probability models quantified by posterior PDF, which are those models most consistent with the experimental data and prior information. For identification and uncertainty quantification, Ching et al. [7] presented a Gibbs sampling based simulation approach for model updating of linear dynamic systems using modal data consisting of classical normal modes. Cheung and Bansal [8] extended the Gibbs sampling based simulation approach for model updating of a linear dynamic system using modal data consisting of non-classical modes. In all the aforementioned approaches it is assumed that the modal data are obtained using a single setup. However, due to practical limitations which results in a limited number of sensors and acquisition channels, it is a common practice to obtain modal data from several setups each covering a different part of the structure. Modal data from multiple setups pose a problem as mode shapes identified from multiple setups are normalized individually and the scaling factors to form the overall mode shape are not known *a priori*.

The objective of this paper is to propose a Gibbs simulation based approach for Bayesian model updating of a linear dynamic system using the modal data obtained using multiple setups. In the proposed approach, the uncertain parameter space that originally comprises model parameters and parameters defining the probabilistic models of the model prediction errors is extended by introducing additional parameters, that is, mode shapes and scaling factors to get conditional distributions that are easier to handle algorithmically. By treating mode shapes and scaling factors as uncertain parameters, full conditional distributions are obtained that in turn facilitate the use of Gibbs sampling. The proposed approach does not require solving the eigenvalue problem of any structural model or matching of model and experimental modes, and is robust to the dimension of the problem. To demonstrate the effectiveness and efficiency of the proposed method, illustrative examples with simulated data are presented.

2. PROPOSED APPROACH

Let $D \equiv \{D_i : i = 1, ..., R\}$ denote the experimentally obtained modal data from a linear dynamic system where R is the number of setups and where $D_i = \{\hat{\omega}_{i,m,s}, \hat{\psi}_{i,m,s} : m = 1, ..., M, s = 1, ..., S\}$ denote the data from the ith setup consisting of natural frequencies $\hat{\omega}_{i,m,s} \in \mathbb{R}^+$ and partial mode shapes $\hat{\psi}_{i,m,s} \in \mathbb{R}^{n_i}$. For each setup, S is the number of data sets obtained and from each data set M is the number of observed modes. It is assumed that the partial mode shapes are normalized to have unit Euclidean norm, that is, $||\hat{\psi}_{i,m,s}|| = \sqrt{\hat{\psi}^T_{i,m,s}\hat{\psi}_{i,m,s}} = 1$. It is possible, although not necessarily, that some degrees of freedom (dof) are measured in multiple setups. Therefore, the number of distinct measured dof from all setups n_o is less than or equal to the number of dof of the identification model $n_d (n_o \leq n_d)$.

2.1 Identification Model

The identification model is based on n_d-dof linear structural models where the mass matrix $\mathbf{M} \in \mathbb{R}^{n_d \times n_d}$ and stiffness $\mathbf{K} \in \mathbb{R}^{n_d \times n_d}$ matrix are represented as a linear sum of contribution of the corresponding mass and stiffness matrices from the individual prescribed substructures:

$$\mathbf{M}(\boldsymbol{\alpha}) = \mathbf{M}_0 + \sum_{i=1}^{N_\alpha} \alpha_i \mathbf{M}_i \tag{2}$$

$$\mathbf{K}(\boldsymbol{\eta}) = \mathbf{K}_0 + \sum_{i=1}^{N_\eta} \eta_i \mathbf{K}_i \tag{3}$$

where $\boldsymbol{\alpha} = [\alpha_1 \cdots \alpha_{N_\alpha}]^T$ and $\boldsymbol{\eta} = [\eta_1 \cdots \eta_{N_\eta}]^T$ are the mass and stiffness scaling parameters, respectively, and $\boldsymbol{\alpha} = [1 \cdots 1]^T$ and $\boldsymbol{\eta} = [1 \cdots 1]^T$ give the nominal mass and stiffness matrices, respectively. The joint PDF of parameters $[\boldsymbol{\alpha}^T \boldsymbol{\eta}^T]^T$ is to be updated by the data D. Natural frequencies ω_m and mode shapes ψ_m for $m = 1, ..., M$ can be obtained from the solution of the following eigenvalue problem:

$$\left(\omega_m^2 \mathbf{M}(\boldsymbol{\alpha}) - \mathbf{K}(\boldsymbol{\eta})\right) \psi_m = 0 \tag{4}$$

Replacing system natural frequencies with experimentally obtained natural frequencies gives

$$\left(\hat{\omega}_{i,m,s}^2 \mathbf{M}(\boldsymbol{\alpha}) - \mathbf{K}(\boldsymbol{\eta})\right) \psi_m = \boldsymbol{\varepsilon}_{i,m,s} \tag{5}$$

where the system mode shape ψ_m is mathematically related to the experimentally obtained mode shape $\hat{\psi}_{i,m,s}$ through a selection matrix $\mathbf{L}_i \in \mathbb{R}^{n_i \times n_d}$ comprising zeros and ones that maps the n_d model dof to the n_i observed dof:

$$\hat{\psi}_{i,m,s} - c_{i,m} \mathbf{L}_i \psi_m = \mathbf{e}_{i,m,s} \tag{6}$$

In Eq. (6), the scaling factor $c_{i,m}$ relates the measured mode shape $\hat{\psi}_{i,m,s}$ to the model predicted mode shape $\mathbf{L}_i \psi_m$. By definition, the overall mode shape is only known up to a scaling constant. For R partial mode shape obtained using R setups, there are effectively only $(R - 1)$ unknown scaling factors to be updated to form the overall mode shapes. In the present formulation, it is assumed that one of the scaling factors for each mode shape is fixed, for example, $c_{1,m} = 1$ for $m = 1, ..., M$. For the case with $R = 1$, Eq. (6) reduces to an equation similar to that defined in Ching et al. [7] for the special case where the modal data are obtained using a single setup.

In Eqs. (5) and (6), $\boldsymbol{\varepsilon}_{i,m,s}$ and $\mathbf{e}_{i,m,s}$ are the random vectors representing the model prediction errors, that is, the errors between the response of the system under consideration and that of the assumed model. The PDFs for vectors $\boldsymbol{\varepsilon}_{i,m,s}$ and $\mathbf{e}_{i,m,s}$ are taken to be Gaussian based on the Principle of Maximum Entropy [21]. Their means are assumed to be equal to zero and covariance matrices equal to scaled versions of the identity matrix $\mathbf{I}$ of appropriate order, respectively:

$$\boldsymbol{\varepsilon}_{i,m,s} \sim N(0, \sigma_m^2 \mathbf{I}) \tag{7}$$

$$\mathbf{e}_{i,m,s} \sim N(0, \delta_{i,m}^2 \mathbf{I}) \tag{8}$$

The overall mode shapes $\psi_m : m = 1, ..., M$, scaling factors $c_{i,m} : i = 2, ..., R, m = 1, ..., M$, and prediction error variances $\sigma_m^2 : m = 1, ..., M$ are not know *a priori* and are introduced as extra parameters to be updated using the system data. Introducing $\delta_{i,m}^2 : i = 1, ..., R; m = 1, ..., M$ as uncertain parameters results in Pareto optimal structural models [22]. To handle this problem, the error variances $\delta_{i,m}^2$ are assumed to be known or are directly estimated from the sample variance of the experimental modal data as follows:

$$\hat{\delta}_{i,m}^2 = \frac{1}{S n_i} \sum_{s=1}^{S} ||\hat{\psi}_{i,m,s} - \bar{\psi}_{i,m}||^2 \tag{9}$$

where $\bar{\psi}_{i,m} = \sum_{s=1}^{S} \hat{\psi}_{i,m,s}/S$ is the averaged mode shape for the mth mode identified in the ith setup. In the proposed Gibbs sampling based approach, four groups of parameters are considered:

$$\theta_1 = [\boldsymbol{\alpha}^T \boldsymbol{\eta}^T]^T \in \mathbb{R}^{N_1}$$
$$\theta_2 = [c_{2,1} \cdots c_{R,M}]^T \in \mathbb{R}^{(R-1)M}$$
$$\theta_3 = [\psi_1^T \cdots \psi_M^T]^T \in \mathbb{R}^{Mn_d}$$
$$\theta_4 = [\sigma_1^2 \cdots \sigma_M^2]^T \in \mathbb{R}^{M}$$

2.2 Prior PDF

The prior PDF for mass and stiffness contribution parameters θ_1 is taken to be a Gaussian PDF, that is, $\theta_1 \sim N(\boldsymbol{\mu}_1^{(0)}, \mathbf{P}_1^{(0)})$, with $\boldsymbol{\mu}_1^{(0)}$ as the most probable values and $\mathbf{P}_1^{(0)}$ as a prior covariance matrix to express the initial uncertainties. For scaling factors θ_2 it is assumed that they can take any real value. The prior PDF for the overall mode shapes θ_3 is taken to be the product of independent uniform PDFs. For the case where any prior information is available to assign a prior PDF for θ_3, the product of independent Gaussian PDFs may be used as the prior PDF. The prior PDF for the prediction error variances θ_4 is taken to be the product of independent inverse gamma PDFs. An inverse gamma PDF $IG(\rho_0,\kappa_0)$ of, e.g., σ^2 with prespecified parameters ρ_0 and κ_0 is

$$p(\sigma^2) = \frac{\kappa_0^{\rho_0}}{\Gamma(\rho_0)} \left(\sigma^2\right)^{-\rho_0-1} \exp\left(-\frac{\kappa_0}{\sigma^2}\right) \tag{10}$$

where the shape parameter ρ_0 and the rate parameter κ_0 are expressed in terms of the mean and coefficient of variation (c.o.v) of σ^2. These choices of the prior PDFs (Bayesian conjugate priors [23]) allow direct sampling from the full conditional $p(\theta_1|\theta_2, \theta_3, \theta_4, D, \mathcal{M})$, $p(\theta_2|\theta_1, \theta_3, \theta_4, D, \mathcal{M})$, $p(\theta_3|\theta_1, \theta_2, \theta_4, D, \mathcal{M})$, and $p(\theta_4|\theta_1, \theta_2, \theta_3, D, \mathcal{M})$ that in turn facilitate the use of Gibbs sampling.

3. GIBBS SAMPLING

The Gibbs sampling [24] is a particular form of MCMC algorithm for obtaining samples from an arbitrary multivariate PDF when direct sampling from the joint distribution is difficult, but sampling from the conditional distributions of each variable, or set of variables, is possible. In the current problem for implementation of the Gibbs sampling to obtain posterior samples distributed as $p(\theta|D, \mathcal{M})$, full conditional PDFs $p(\theta_1|\theta_2, \theta_3, \theta_4, D, \mathcal{M})$, $p(\theta_2|\theta_1, \theta_3, \theta_4, D, \mathcal{M})$, $p(\theta_3|\theta_1, \theta_2, \theta_4, D, \mathcal{M})$, and $p(\theta_4|\theta_1, \theta_2, \theta_3, D, \mathcal{M})$ are required.

3.1 Conditional PDF $p(\theta_1|\theta_2, \theta_3, \theta_4, D, \mathcal{M})$

Given $\theta_2, \theta_3, \theta_4$, and D, Eq. (5) is linear with respect to θ_1 and can be written in the following form:

$$\mathbf{Y}_1 - \mathbf{A}_1\theta_1 = \mathrm{E}_1 \tag{11}$$

where $\mathbf{Y}_1$ and $\mathbf{A}_1$ are given as follows:

$$\mathbf{Y}_1 = \left[\mathbf{b}_{1,1,1}^T \cdots \mathbf{b}_{i,m,s}^T \cdots \mathbf{b}_{R,M,S}^T\right]^T \in \mathbb{R}^{n_d RMS \times 1}$$
$$\mathbf{b}_{i,m,s} = \left(\hat{\omega}_{i,m,s}^2 \mathbf{M}_0 - \mathbf{K}_0\right) \psi_m \in \mathbb{R}^{n_d \times 1}$$
$$\mathbf{A}_1 = -\left[\mathbf{d}_{1,1,1}^T \cdots \mathbf{d}_{i,m,s}^T \cdots \mathbf{d}_{R,M,S}^T\right]^T \in \mathbb{R}^{n_d RMS \times N_1}$$
$$\mathbf{d}_{i,m,s} = \left[\hat{\omega}_{i,m,s}^2 \mathbf{M}_1 \psi_m \cdots \hat{\omega}_{i,m,s}^2 \mathbf{M}_{N_\alpha} \psi_m \;\; \mathbf{K}_1 \psi_m \cdots \mathbf{K}_{N_\eta} \psi_m\right] \in \mathbb{R}^{n_d \times N_1}$$

$\mathbf{E}_1$ follows a Gaussian distribution with zero mean and covariance matrix given by

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} \sigma_1^2\mathbf{I} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \sigma_M^2\mathbf{I} \end{bmatrix} \in \mathbb{R}^{n_d RMS \times n_d RMS} \tag{12}$$

$\mathbf{Y}_1$ and $\mathbf{A}_1$ are fixed matrices given θ_3 and D, and $\boldsymbol{\Sigma}_1$ is a fixed covariance matrix given θ_4. Then, the full conditional PDF $p(\theta_1|\theta_2, \theta_3, \theta_4, D, \mathcal{M})$ is a Gaussian distribution with mean $\boldsymbol{\mu}_1$ and covariance matrix $\mathbf{P}_1$ given by

$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_1^{(0)} + \mathbf{P}_1^{(0)}\mathbf{A}_1^T\left(\boldsymbol{\Sigma}_1 + \mathbf{A}_1\mathbf{P}_1^{(0)}\mathbf{A}_1^T\right)^{-1}\left(\mathbf{Y}_1 - \mathbf{A}_1\boldsymbol{\mu}_1^{(0)}\right) \tag{13}$$

$$\mathbf{P}_1 = \mathbf{P}_1^{(0)} - \mathbf{P}_1^{(0)}\mathbf{A}_1^T\left(\boldsymbol{\Sigma}_1 + \mathbf{A}_1\mathbf{P}_1^{(0)}\mathbf{A}_1^T\right)^{-1}\mathbf{A}_1\mathbf{P}_1^{(0)} \tag{14}$$

3.2 Conditional Distribution $p(\theta_2|\theta_1, \theta_3, \theta_4, D, \mathcal{M})$

For each mode, the model prediction errors $\delta_{i,m}^2$ are assumed to be independent. Therefore, scaling factors $\theta_{2,m} = [c_{2,m} \cdots c_{R,m}]^T$ for each mode shape are updated independently. Given θ_3 and D, Eq. (6) is linear with respect to $\theta_{2,m}$ and can be written in the following form:

$$\mathbf{Y}_{2,m} - \mathbf{A}_{2,m}\theta_{2,m} = \mathbf{E}_{2,m} \tag{15}$$

where $\mathbf{Y}_{2,m}$ and $\mathbf{A}_{2,m}$ are given as follows:

$$\mathbf{Y}_{2,m} = [\hat{\psi}_{2,m,1}^T \ \cdots \hat{\psi}_{i,m,s}^T \ \cdots \hat{\psi}_{R,m,S}^T]^T \in \mathbb{R}^{\sum_{i=2}^{R} n_i S \times 1}$$

$$\mathbf{A}_{2,m} = \begin{bmatrix} \mathbf{L}_2\psi_m & & & & \mathbf{0} \\ & \ddots & & & \\ & & \mathbf{L}_i\psi_m & & \\ & & & \ddots & \\ \mathbf{0} & & & & \mathbf{L}_R\psi_m \end{bmatrix} \in \mathbb{R}^{\sum_{i=2}^{R} n_i S \times (R-1)}$$

$\mathbf{E}_{2,m}$ follows a Gaussian distribution with zero mean and covariance matrix:

$$\boldsymbol{\Sigma}_{2,m} = \begin{bmatrix} \delta_{2,m}^2\mathbf{I} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \delta_{R,m}^2\mathbf{I} \end{bmatrix} \in \mathbb{R}^{\sum_{i=2}^{R} n_i S \times \sum_{i=2}^{R} n_i S} \tag{16}$$

$\mathbf{Y}_{2,m}$, $\mathbf{A}_{2,m}$, and $\boldsymbol{\Sigma}_{2,m}$ are fixed matrices given θ_3 and D. Then, the conditional PDF $p(\theta_{2,m}|\theta_1, \theta_3, \theta_4, D, \mathcal{M})$ is a Gaussian distribution with mean $\boldsymbol{\mu}_{2,m}$ and covariance matrix $\mathbf{P}_{2,m}$ given by

$$\boldsymbol{\mu}_{2,m} = \left(\mathbf{A}_{2,m}^T\boldsymbol{\Sigma}_{2,m}^{-1}\mathbf{A}_{2,m}\right)^{-1}\mathbf{A}_{2,m}^T\boldsymbol{\Sigma}_{2,m}^{-1}\mathbf{Y}_{2,m} \tag{17}$$

$$\mathbf{P}_{2,m} = \left(\mathbf{A}_{2,m}^T\boldsymbol{\Sigma}_{2,m}^{-1}\mathbf{A}_{2,m}\right)^{-1} \tag{18}$$

3.3 Conditional Distribution $p(\theta_3|\theta_1, \theta_2, \theta_4, D, \mathcal{M})$

The full conditional PDF $p(\theta_3|\theta_1, \theta_2, \theta_4, D, \mathcal{M})$ can be written as follows:

$$p(\theta_3|\theta_1, \theta_2, \theta_4, D, \mathcal{M}) = \prod_{m=1}^{M} p(\theta_{3,m}|\theta_1, \theta_2, \theta_4, D, \mathcal{M}) \tag{19}$$

Therefore, each mode shape $\theta_{3,m} : m = 1, ..., M$ is updated independently. Conditioned on $\theta_1, \theta_2, \theta_4$, and D, Eqs. (5) and (6) can be written in the following form:

$$\mathbf{Y}_{3,m} - \mathbf{A}_{3,m}\theta_{3,m} = \mathbf{E}_{3,m} \tag{20}$$

where $\mathbf{Y}_{3,m}$ and $\mathbf{A}_{3,m}$ are given as follows:

$$\mathbf{Y}_{3,m} = [\mathbf{0}^{1\times n_d} \;\; \hat{\psi}^T_{1,m,1} \;\cdots\; \mathbf{0}^{1\times n_d} \;\; \hat{\psi}^T_{i,m,s} \;\cdots\; \mathbf{0}^{1\times n_d} \;\; \hat{\psi}^T_{R,m,S}]^T \in \mathbb{R}^{\left(n_d R + \sum\limits_{i=1}^{R} n_i\right)S\times 1}$$

$$\mathbf{A}_{3,m} = \begin{bmatrix} \hat{\omega}^2_{1,m,1}\mathbf{M} - \mathbf{K} \\ c_{1,m}\mathbf{L}_1 \\ \vdots \\ \hat{\omega}^2_{i,m,s}\mathbf{M} - \mathbf{K} \\ c_{i,m}\mathbf{L}_i \\ \vdots \\ \hat{\omega}^2_{R,m,S}\mathbf{M} - \mathbf{K} \\ c_{R,m}\mathbf{L}_R \end{bmatrix} \in \mathbb{R}^{\left(n_d R + \sum\limits_{i=1}^{R} n_i\right)S\times n_d}$$

$\mathbf{E}_{3,m}$ follows a Gaussian distribution with zero mean and covariance matrix:

$$\boldsymbol{\Sigma}_{3,m} = \begin{bmatrix} \sigma^2_m\mathbf{I} & & & & \mathbf{0} \\ & \delta^2_{1,m}\mathbf{I} & & & \\ & & \ddots & & \\ & & & \sigma^2_m\mathbf{I} & \\ \mathbf{0} & & & & \delta^2_{R,m}\mathbf{I} \end{bmatrix} \in \mathbb{R}^{\left(n_d R + \sum\limits_{i=1}^{R} n_i\right)S\times\left(n_d R + \sum\limits_{i=1}^{R} n_i\right)S} \tag{21}$$

$\mathbf{Y}_{3,m}$, $\mathbf{A}_{3,m}$, and $\boldsymbol{\Sigma}_{3,m}$ are fixed matrices given θ_1, θ_2, θ_4, and D. Then, the conditional PDF $p(\theta_{3,m}|\theta_1, \theta_2, \theta_4, D, \mathcal{M})$ is a Gaussian distribution with mean $\boldsymbol{\mu}_{3,m}$ and covariance matrix $\mathbf{P}_{3,m}$ given by

$$\boldsymbol{\mu}_{3,m} = \left(\mathbf{A}^T_{3,m}\boldsymbol{\Sigma}^{-1}_{3,m}\mathbf{A}_{3,m}\right)^{-1}\mathbf{A}^T_{3,m}\boldsymbol{\Sigma}^{-1}_{3,m}\mathbf{Y}_{3,m} \tag{22}$$

$$\mathbf{P}_{3,m} = \left(\mathbf{A}^T_{3,m}\boldsymbol{\Sigma}^{-1}_{3,m}\mathbf{A}_{3,m}\right)^{-1} \tag{23}$$

3.4 Conditional Distribution $p(\theta_4|\theta_1, \theta_2, \theta_3, D, \mathcal{M})$

With the previous construction, the posterior PDFs of prediction error variances given $\theta_1, \theta_2, \theta_3$, and D are independent. Thus, $p(\sigma^2_m|\theta_1, \theta_2, \theta_3, D, \mathcal{M})$ for $m = 1, ..., M$ are inverse gamma given as follows:

$$p(\sigma^2_m|\theta_1, \theta_2, \theta_3, D, \mathcal{M}) = IG\left(\rho_0 + \frac{n_d RS}{2}, \kappa_0 + \frac{1}{2}\sum_{i=1}^{R}\sum_{s=1}^{S}\boldsymbol{\varepsilon}^T_{i,m,s}\boldsymbol{\varepsilon}_{i,m,s}\right) \tag{24}$$

3.5 Summary of the Proposed Algorithm

Let $\theta^{(n)}$ denote the nth sample of θ where $\theta = [\theta_1^T\theta_2^T\theta_3^T\theta_4^T]^T$.

1. Draw the starting point $\theta^{(0)}$ from the prior PDF or use nominal values as the starting point and let $n = 0$.
2. Sample $\theta_1^{(n+1)}$ from $p(\theta_1|\theta_2^{(n)}, \theta_3^{(n)}, \theta_4^{(n)}, D, \mathcal{M})$, which is a Gaussian distribution with mean $\boldsymbol{\mu}_1$ and covariance matrix $\mathbf{P}_1$.

3. Sample $\theta_2^{(n+1)}$ by sampling $\theta_{2,m}^{(n+1)}$ from $p(\theta_{2,m}|\theta_1^{(n+1)}, \theta_3^{(n)}, \theta_4^{(n)}, D, \mathcal{M})$, which is a Gaussian distribution with mean $\mu_{2,m}$ and covariance matrix $\mathbf{P}_{2,m}$ for $m = 1, ..., M$.

4. Sample $\theta_3^{(n+1)}$ by sampling $\theta_{3,m}^{(n+1)}$ from $p(\theta_{3,m}|\theta_1^{(n+1)}, \theta_2^{(n+1)}, \theta_4^{(n)}, D, \mathcal{M})$, which is a Gaussian distribution with mean $\mu_{3,m}$ and covariance matrix $\mathbf{P}_{3,m}$ for $m = 1, ..., M$.

5. Sample $\theta_4^{(n+1)}$ by sampling $\sigma_m^{2(n+1)}$ from $p(\sigma_m^2|\theta_1^{(n+1)}, \theta_2^{(n+1)}, \theta_3^{(n+1)}, D, \mathcal{M})$ for $m = 1, ..., M$, which is an inverse gamma distribution.

6. Let $n = n+1$ and go to step 2, until N samples are obtained.

Usually some initial portion of the Gibbs sequence is discarded before the stationary stage is reached. In the present implementation the burn-in period is determined visually by plotting the Markov samples against iteration number. After the burn-in period, the Gibbs sequence converges to a target PDF that is independent of the starting values. Statistical measures such as the mean, variance, or marginal distribution can be estimated using the remaining samples. The Gibbs sampling approach is effective for globally identifiable and unidentifiable problems. For locally identifiable cases, where the regions of high values of the posterior PDF are well separated, the samples obtained using a single run may get trapped in the neighborhood of only one of the optimal models.

When the interest is to calculate the most probable model, it can be obtained by iteratively optimizing the conditional PDFs $p(\theta_1|\theta_2, \theta_3, \theta_4, D, \mathcal{M})$, $p(\theta_2|\theta_1, \theta_3, \theta_4, D, \mathcal{M})$, $p(\theta_3|\theta_1, \theta_2, \theta_4, D, \mathcal{M})$, and $p(\theta_4|\theta_1, \theta_2, \theta_3, D, \mathcal{M})$ with respect to θ_1, θ_2, θ_3, and θ_4, respectively. For Gaussian conditional PDF of θ_1, θ_2, and θ_3 given the other parameters fixed at the most current values (shown in Sections 3.1–3.3, respectively), the corresponding optimal point for θ_1, θ_2, and θ_3 is given by the corresponding conditional mean given by Eqs. (13), (17), and (22), respectively. The optimal point of an inverse gamma conditional PDF of θ_4 given the other parameters fixed at the most current values (shown in Section 3.4) is given by $\rho/(\kappa + 1)$, where ρ and κ are given by the first and the second numbers in the parentheses in Eq. (24), respectively.

4. ILLUSTRATIVE EXAMPLE

4.1 Example 1

For the first example, consider a 10-dof shear building system with one horizontal dof at each floor, and story masses $\tilde{m}_1 = \cdots = \tilde{m}_{10} =$ 1,000 and interstory stiffnesses $\tilde{k}_1 = \cdots = \tilde{k}_{10} =$ 8,000. The modal data consist of data from three setups ($R = 3$). Assume only three sensors are available due to practical limitations. A setup plan is systematically shown in Fig. 1. In the figure, the squares indicate the sensor location. Data from the first setup consist of natural frequencies and mode shape components corresponding to dof one, two, and three ($n_1 = 3$). Data from the second setup consist of natural frequencies and mode shape components corresponding to dof five, six, and seven ($n_2 = 3$). And, data from the third setup consist of natural frequencies and mode shape components corresponding to dof eight, nine, and ten ($n_3 = 3$). The total number of dof observed is equal to nine from the three setups. Data from each setup consist of four sets of simulated modal data ($S = 4$) with the first three modal frequencies and partial mode shapes ($M = 3$). Noisy measured modal data are generated by adding random values chosen from zero-mean Gaussian distribution with standard deviation equal to 3% of the exact values.

For identification and uncertainty quantification, the same 10-dof shear building system is considered with the mass and stiffness matrices parameterized as follows: $m_i = \alpha_i \tilde{m}_i$, $k_i = \eta_i \tilde{k}_i$, $i = 1, 2, ..., 10$. The uncertain parameters whose joint PDF is to be updated for this model class are mass and stiffness scaling parameters $[\alpha_1 \cdots \alpha_{10}]^T$, $[\eta_1 \cdots \eta_{10}]^T$, overall mode shapes for the first three modes $[\psi_1^T \cdots \psi_3^T]^T$, scaling factors $[c_{2,1} \cdots c_{3,3}]^T$, and prediction error variances $[\sigma_1^2 \cdots \sigma_3^2]^T$. Smaller values of c.o.v are assumed for mass parameters since these parameters can usually be determined more accurately from the structural drawings than other parameters. Their prior PDFs are chosen to be Gaussian with mean values equal to 1 and the c.o.v for each equal to 1%. For stiffness parameters that are not very well known *a priori*, relatively larger c.o.v is assumed. Their prior PDFs are chosen to be Gaussian with mean values equal to 1 and the c.o.v for each equal to 20%. For scaling factors, it is assumed that they can take any

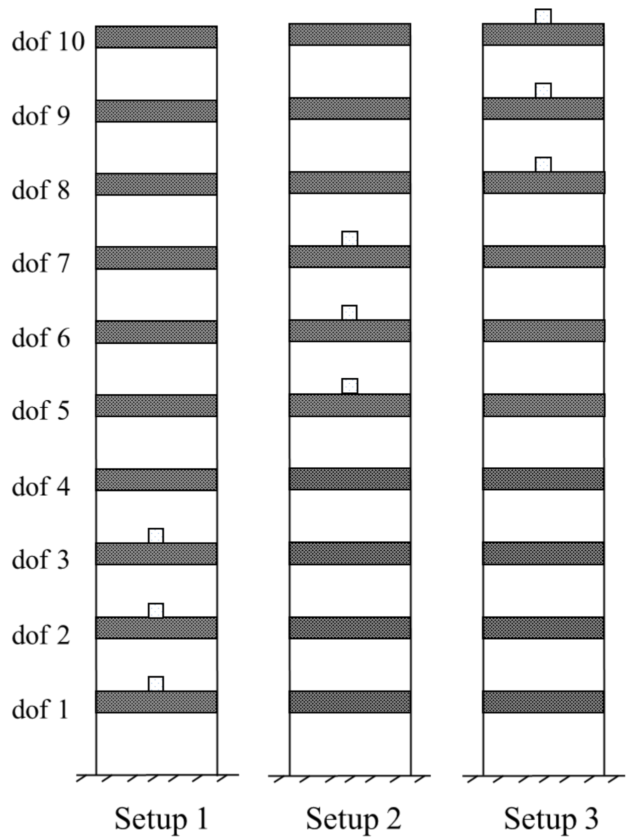

FIG. 1: 10-dof shear building system.

real value. Flat independent priors are taken for mode shape components. Independent inverse gamma prior PDFs with $\rho_0 = 0$, $\kappa_0 = 0$ are taken for prediction error variances (Jeffreys' non-informative prior). The total number of uncertain parameters whose joint PDF is to be updated is 59 (10 for mass scaling parameters, 10 for stiffness scaling parameters, 6 for scaling factors, 30 for three mode shapes, and 3 for prediction error variances).

Following the proposed Gibbs sampling based algorithm, Markov chain samples of mass and stiffness scaling parameters, mode shapes, scaling factors, and prediction error variances are obtained. The starting point of the Gibbs sampling is simulated from the prior PDFs of the uncertain parameters. In order to confirm that the Markov chain had converged to the stationary distribution, 10 independent simulations were performed. The resulting numerical results were consistent across these simulations, suggesting that the Markov chain is ergodic.

Figure 2 shows the Markov chain samples of some stiffness scaling parameters from one of the Markov chains. The dashed line indicates the end of the burn-in period. Figure 3 shows the posterior mean estimates of mode shapes (solid curve) and actual system mode shapes (dotted curve) corresponding to the structural model that generated the data for the first three modes. The two sets are quite close to each other, indicating the effectiveness of the proposed approach to identify the overall mode shapes.

Table 1 shows some statistical properties of the posterior marginal PDFs of the mass and stiffness scaling parameters estimated using the posterior samples. It shows actual system values (column 2), posterior mean (column 3), posterior standard deviation (column 4), and posterior c.o.v (column 5). It can be seen that the posterior mean estimates are close to those corresponding to the actual system and the posterior c.o.v estimates are much smaller than those that were initially assumed since the data provide information about these parameters. Table 2 shows the actual system scaling factor corresponding to the partial mode shapes observed in different setups and the posterior mean estimates of the scaling factor samples. It can be seen that the posterior mean estimates are close to the values

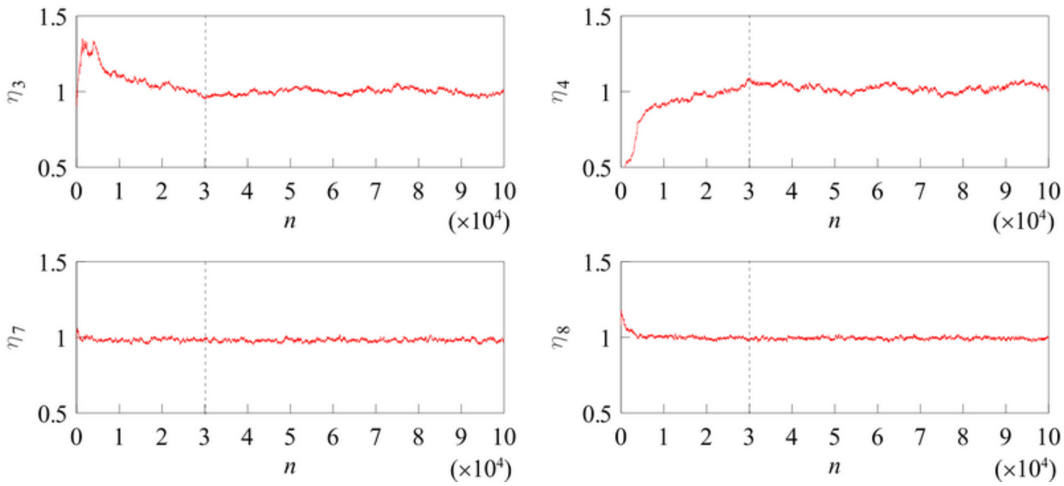

FIG. 2: Markov chain samples for the stiffness scaling parameters.

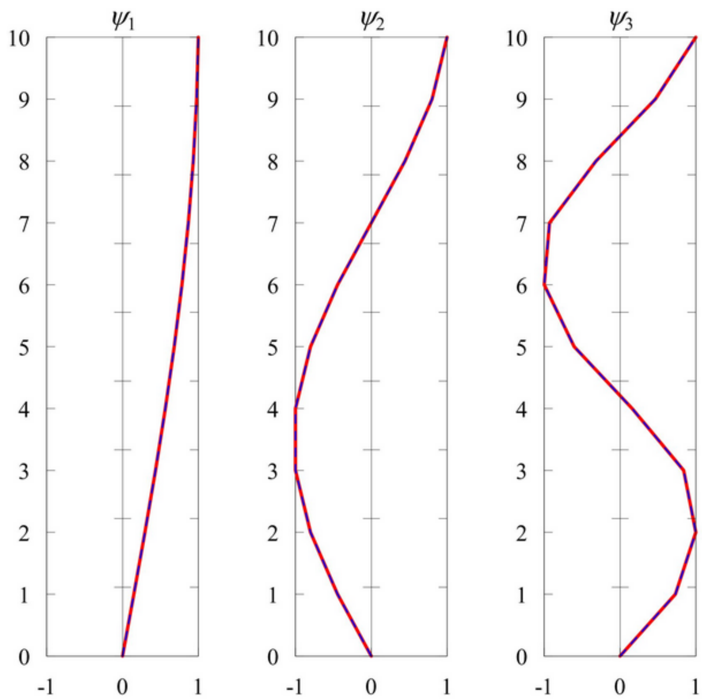

FIG. 3: Posterior mean estimates of mode shapes (solid curve) and actual system mode shapes (dotted curve).

from the actual system, once again indicating the effectiveness of the proposed approach to identify the overall mode shapes.

4.2 Example 2

The system selected for this example is a three-story, three-bay by three-bay steel frame FE model (shown in Fig. 4) corresponding to example 8 of the illustrative examples for the non-linear FE software OpenSees [25]. The detailed description of the FE model can be obtained from the website (http://opensees.berkeley.edu/wiki/index.php/

TABLE 1: Statistical properties of the posterior mass and stiffness scaling parameter samples

Parameter	Actual values	Posterior mean	Posterior std. dev.	Posterior c.o.v
η_1	1.000	0.995	0.010	0.010
η_2	1.000	0.983	0.007	0.008
η_3	1.000	0.998	0.020	0.020
η_4	1.000	1.024	0.025	0.024
η_5	1.000	0.995	0.010	0.010
η_6	1.000	0.994	0.009	0.009
η_7	1.000	0.981	0.011	0.011
η_8	1.000	0.993	0.009	0.009
η_9	1.000	0.996	0.009	0.009
η_{10}	1.000	1.000	0.007	0.007
α_1	1.000	0.993	0.009	0.009
α_2	1.000	0.995	0.009	0.009
α_3	1.000	1.005	0.010	0.010
α_4	1.000	1.006	0.009	0.009
α_5	1.000	1.003	0.009	0.009
α_6	1.000	0.982	0.009	0.009
α_7	1.000	0.993	0.009	0.009
α_8	1.000	0.992	0.008	0.009
α_9	1.000	1.005	0.008	0.008
α_{10}	1.000	0.992	0.009	0.009

TABLE 2: Statistical properties of the posterior mode shape scaling factor samples

Setup	Mode	Actual system	Posterior mean
$i = 1$	$m = 1$	1.000	1.00 (fixed)
	$m = 2$	1.000	1.00 (fixed)
	$m = 3$	1.000	1.00 (fixed)
$i = 2$	$m = 1$	1.423	1.418
	$m = 2$	1.000	1.004
	$m = 3$	0.941	0.956
$i = 3$	$m = 1$	3.075	3.082
	$m = 2$	1.000	0.999
	$m = 3$	0.743	0.753

Examples_Manual). The general properties of the model are briefly described here. It has a 21.95 m × 21.95 m plan and is 12.80 m tall. Columns and beams have W27 × 114 and W24 × 94 steel sections, respectively, and each floor has a 15.24 cm thick slab. The x direction is the weak direction of the columns. This model is termed as the nominal FE model.

Modal data for identification and uncertainty quantification are generated using the nominal FE model. The simulated modal data consist of data from two setups ($R = 2$). In the first setup, mode shape components corresponding to dof one and two are available ($n_1 = 2$). In the second setup, mode shape components corresponding to dof two and three are available ($n_2 = 2$). Data from each setup consist of 10 sets of modal data ($S = 10$) with the first translational mode observed in the weak ($x-$) direction ($M = 1$). Noisy measured modal data are generated by adding random values chosen from zero-mean Gaussian distribution with standard deviation equal to 2% of the exact values.

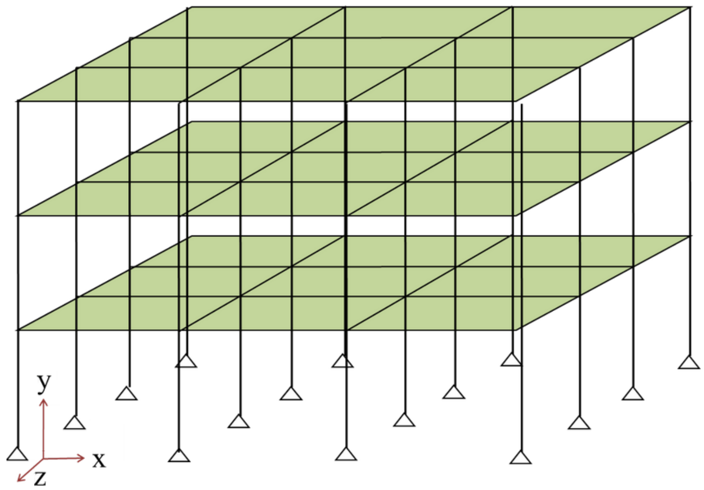

FIG. 4: Nominal FE model.

For identification and uncertainty quantification, a two-dimensional frame model is used to represent the behavior of the FE model in the x direction (shown in Fig. 5). Each floor has one translational dof (along the x direction) and four rotational dof (along the z axis). Translation along the y direction is not allowed in the identification process. The mass for each story is lumped at the floor level. The mass and stiffness matrices for the model are parameterized as follows:

$$\mathbf{M}(\alpha) = \sum_{i=1}^{3} \alpha_i \mathbf{M}_i \tag{25}$$

$$\mathbf{K}(\eta) = \sum_{i=1}^{3} \eta_i \mathbf{K}_i \tag{26}$$

where $\mathbf{M}_i$ and $\mathbf{K}_i$ are the mass and stiffness sub-matrices, respectively, defined for each story and are estimated using the properties of the nominal FE model. The assumed structural model for identification and uncertainty quantification is not equivalent to the nominal FE model used to simulate the modal data.

Masses are assumed to be known with small uncertainty. Thus, the prior PDF for $[\alpha_1\, \alpha_2\, \alpha_3]^T$ is assumed to be independent Gaussian with mean values equal to 1 and c.o.v equal to 1%. The prior PDF for $[\eta_1\, \eta_2\, \eta_3]^T$ is assumed to be independent Gaussian with mean values equal to 1 and c.o.v equal to 10%. Flat independent priors are taken for mode shapes and scaling factors, and a product of independent inverse gamma non-informative prior PDFs is taken for

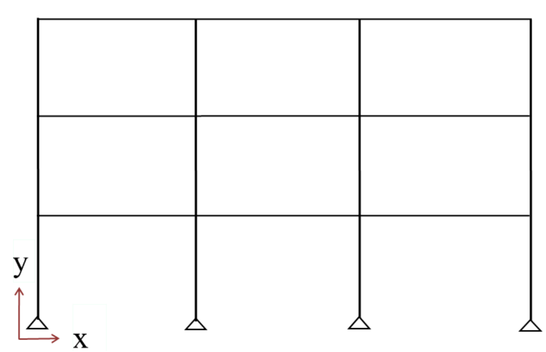

FIG. 5: Structural model to represent the behavior of the nominal FE model in the x direction.

prediction error variances. In total, there are 27 uncertain parameters to be updated (6 for mass and stiffness scaling parameters, 19 for mode shape, 1 for scaling factor, and 1 for prediction error variance).

Following the proposed Gibbs sampling based algorithm, 50,000 samples of mass and stiffness scaling parameters, mode shapes, scaling factors, and prediction errors variances are obtained. Figure 6 shows the Markov chain samples of stiffness scaling parameters. The dashed line indicates the end of the burn-in period. Figure 7 shows prior and posterior stiffness scaling samples projected onto two-dimensional space. The figure shows that the uncertainty for the stiffness scaling parameters is reduced considerably when the information contained in the data is used. Table 3 shows some statistical properties of the posterior marginal PDFs of the mass and stiffness scaling parameters estimated using the posterior samples. It can be seen that the posterior mean estimates are close to those corresponding to the nominal system and the posterior c.o.v estimates are much smaller than those that were initially assumed. The posterior c.o.v estimate for the first story is smaller than the posterior c.o.v estimate for the second and third story because the first natural frequency is much more sensitive to the first story stiffness and the modal data consist of only first translational mode.

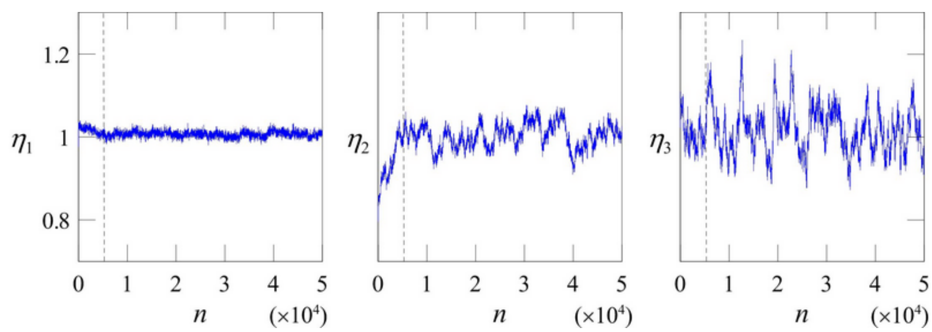

FIG. 6: Markov chain samples for the stiffness scaling parameters.

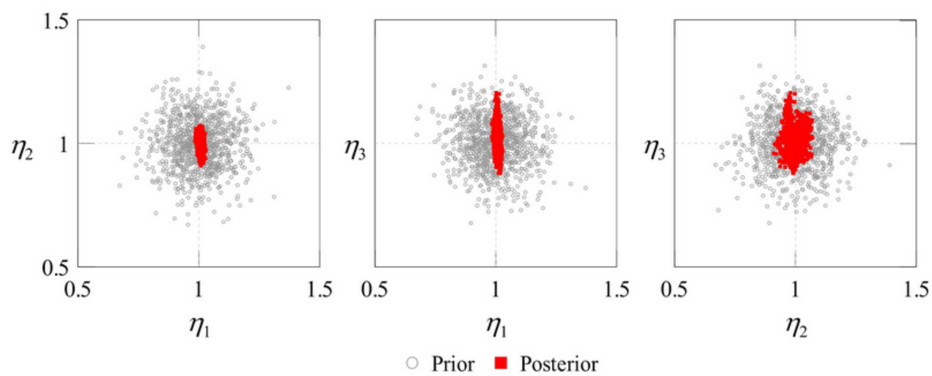

FIG. 7: Prior and posterior samples for pairs $\{\eta_1\,\eta_2\}$, $\{\eta_1\,\eta_3\}$, and $\{\eta_2\,\eta_3\}$.

TABLE 3: Statistical properties of the posterior mass and stiffness scaling parameter samples

Parameter	**Posterior mean**	**Posterior std. dev.**	**Posterior c.o.v**
η_1	1.006	0.008	0.008
η_2	1.001	0.030	0.030
η_3	1.015	0.055	0.054
α_1	0.991	0.010	0.010
α_2	0.987	0.010	0.010
α_3	0.984	0.010	0.010

5. CONCLUSIONS

In this work, a new Gibbs sampling based approach for Bayesian model updating of a linear dynamic system based on incomplete modal data (modal frequencies and partial mode shapes of some of the dominant modes) obtained from a structure using multiple setups is proposed. The proposed approach does not require solving the eigenvalue problem of any structural model or matching of model and experimental modes. The results from the illustrative examples demonstrate that the posterior samples for the uncertain parameters are reasonable when compared with the nominal system values, indicating the effectiveness of the procedure. The proposed method allows the uncertainty of the parameters to be updated efficiently even if there are a large number of uncertain parameters. The case considering the non-classical modes is left for future work.

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