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SOME A PRIORI ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS OF ELLIPTIC AND PARABOLIC LINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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We study some theoretical aspects of Legendre polynomial chaos based finite element approximations of elliptic and parabolic linear stochastic partial differential equations (SPDEs) and provide a priori error estimates in tensor product Sobolev spaces that hold under appropriate regularity assumptions. Our analysis takes place in the setting of finite-dimensional noise, where the SPDE coefficients depend on a finite number of second-order random variables. We first derive a priori error estimates for finite element approximations of a class of linear elliptic SPDEs. Subsequently, we consider finite element approximations of parabolic SPDEs coupled with a θ-weighted temporal discretization scheme. We establish conditions under which the time-stepping scheme is stable and derive a priori rates of convergence as a function of spatial, temporal, and stochastic discretization parameters. We later consider steady-state and time-dependent stochastic diffusion equations and illustrate how the general results provided here can be applied to specific SPDE models. Finally, we theoretically analyze primal and adjoint-based recovery of stochastic linear output functionals that depend on the solution of elliptic SPDEs and show that these schemes are superconvergent.

KEY WORDS: *stochastic partial differential equations, a priori error estimation, chaos expansions, finite element methods, time-stepping stability, functional approximation*

1. INTRODUCTION

Since the publication of the monograph on spectral stochastic finite element (FE) methods by Ghanem and Spanos [1], a number of researchers have developed and applied numerical schemes based on this idea with great success to a broad range of stochastic partial differential equations (SPDEs); see, for example [2–6]. Due to the increasing popularity of stochastic FE methods, there has been a growing interest in theoretical analysis of this class of numerical schemes in order to derive a priori rates of convergence and error estimates. Such results can provide valuable insights into stochastic FE methods and are also of practical importance for computational implementations.

Stochastic diffusion models have been extensively studied from a theoretical and numerical point of view in the literature since it is a representative SPDE model widely used to study the performance of numerical methods for SPDEs; see, for example [7–15]. Some convergence rates in tensor product Sobolev spaces for the FE approximations of such models are provided in [7–9, 11, 15] when considering steady-state stochastic diffusion models and in [13] when considering time-dependent stochastic diffusion models. The convergence rate of best N-term approximations has been studied by Cohen et al. [11] under appropriate assumptions on the Karhunen-Loève (KL) expansion of the random field. When the SPDE solution satisfies appropriate analyticity conditions in the complex plane, exponential convergence rates have been proved for the same kind of models by Nobile, Tempone, and co-workers [9, 13].

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NOMENCLATURE

$\mathcal{D}$	bounded convex physical domain, $\mathcal{D} \subset \mathbb{R}^d$
$\partial\mathcal{D}$	polygonal boundary of the physical domain
$\mathbf{x}$	spatial coordinates
T	integration time, $T < +\infty$
t	temporal coordinate, $t \in [0, T]$
Ω	sample space, $\Omega \subset \mathbb{R}^q$
ω	elementary event, $\omega \in \Omega$
$\boldsymbol{\xi}$	vector of M independent uniform random variables, $\boldsymbol{\xi} : \Omega \to \mathbb{R}^M$
Γ	joint image of $\boldsymbol{\xi}$, $\Gamma \subset \mathbb{R}^M$
$\rho(\boldsymbol{\xi})$	probability density function
$\langle\cdot\rangle$	expectation operator with respect to ρ
$\mathcal{V}$	Hilbert space of spatial functions
$\mathcal{S}$	Hilbert space of random functions
$\mathcal{W}$	tensor product space, $\mathcal{W} = \mathcal{V} \otimes \mathcal{S}$
u	solution of SPDE model, $u \in \mathcal{W}$ (elliptic case), $u \in L^2(0, T; \mathcal{W})$ (parabolic case)
u_0	initial value function (parabolic case)
A	bilinear form in weak formulations
α_c	continuity constant of A
α_e	ellipticity constant of A, $\alpha_e > 0$
$\mathscr{T}_h$	nondegenerate or quasi-uniform triangulation of $\mathcal{D}$
h	mesh-size, $h \in [0, 1[$
$\mathcal{V}^h$	finite-dimensional subspace, $\mathcal{V}^h \subset \mathcal{V}$
n	number of spatial degrees of freedom
PC	polynomial chaos
$\mathcal{S}^{p_\xi}$	M-dimensional PC space, $\mathcal{S}^{p_\xi} \subset \mathcal{S}$
p_ξ	PC degree
N_ξ	cardinality of $\mathcal{S}^{p_\xi}$
u_{h,p_ξ}	approximate solution in $\mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$ (elliptic case)
θ	parameter of the weighted temporal discretization scheme, $\theta \in [0, 1]$
Δt	time-step
t^m	instant time $t^m = m\Delta t$
u^m_{h,p_ξ}	approximate solution in $\mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$ at time t^m (parabolic case)
$J(u)$	linear functional depending on the solution u of the primal problem, $u \in \mathcal{V} \otimes \mathcal{S}$
$J(u_{h,p_\xi})$	approximate linear functional
w	solution of the dual problem, $w \in \mathcal{V} \otimes \widetilde{\mathcal{S}}$
H	(coarse) mesh-size for the spatial discretization of the dual problem
q_ξ	(low) PC order for the stochastic discretization of the dual problem
w_{H,q_ξ}	approximate dual solution in $\mathcal{V}^H \otimes \mathcal{S}^{q_\xi}$
$J_{imp}(u_{h,p_\xi}, w_{H,q_\xi})$	approximate improved linear functional
$C_{\mathcal{D}}$	Poincaré's constant

Convergence rates with respect to Sobolev norms have also been studied by Todor and Schwab [15] when considering sparse Wiener-chaos approximations.

Recently, Bespalov et al. [16] provided a detailed a priori error analysis for stochastic Galerkin mixed approximations of elliptic SPDEs. Mugler and Starkloff [17] proposed a new approach based on a stochastic Petrov-Galerkin projection scheme for solving the steady-state stochastic diffusion equation with boundedness assumptions on the random coefficients weaker than those usually considered in the literature. Qu and Xu [18] presented convergence analysis of a stochastic Galerkin approach for solving the Stokes equations with random coefficients, whose solution is discretized using spectral and generalized polynomial chaos (PC) expansions for its spatial and random part, respectively. In particular, the analysis of Babuška et al. [7] for stochastic elliptic SPDEs is extended to saddle-point problems. Error estimates (in classical Bochner spaces) for the spatial FE approximation of the steady-state diffusion equation with log-normal random coefficients are derived in [19], including KL truncation and quadrature errors.

In the case of parabolic SPDE models, error estimates have been proved by Nobile and Tempone [13] for the *semi-discrete* model, i.e., with no temporal discretization. These estimates hold under appropriate analyticity assumptions in the complex plane. We would like to highlight here that there also exists a vast literature in the setting of infinite-dimensional noise; see, for example [20–27]. The focus of the present analysis is on finite-dimensional noise in the setting of tensor product Sobolev spaces.

In the present work, we consider elliptic and parabolic SPDEs in the setting of finite-dimensional noise. We focus on the case when the SPDE coefficients depend on a finite number of independent and identically distributed

(i.i.d) uniform random variables. For example, SPDE coefficients modeled as random fields can be discretized using a truncated KL expansion [28], leading to the SPDE solution depending on a finite number of uncorrelated random variables. For the sake of convenience, statistical independence of the random variables is often introduced as an additional modeling assumption [7, 14]. Our objective is to derive *a priori* error estimates in Sobolev norms for Legendre chaos-based FE approximations of a class of elliptic and parabolic linear SPDEs. The analysis presented here is based on the assumption that the SPDE solution satisfies certain spatial and stochastic regularity conditions. More specifically, we shall assume that the partial derivatives of the SPDE solution with respect to the spatial and stochastic coordinates, up to a given order, are square integrable. Note that this is different from the polydisc analyticity assumptions used in previous theoretical studies on elliptic SPDEs [9] and the semi-discrete form of parabolic SPDEs [13]. We first derive a priori error estimates for FE approximations of elliptic SPDEs. Following this, we present stability analysis of a class of weighted temporal discretization schemes applied to parabolic SPDEs. To the best of our knowledge, stability analysis of temporal discretization schemes for parabolic SPDE models has not been presented before in the literature. Based on the stability analysis results, we prove *a priori* error estimates for FE approximations of parabolic SPDEs (in conjunction with Legendre chaos expansions and weighted temporal discretization) as a function of the spatial, temporal, and stochastic discretization parameters. We also show how sharper a priori error estimates can be derived for elliptic and parabolic SPDE models using duality arguments.

To illustrate how the theoretical results can be applied to specific SPDE models, we consider the steady-state and time-dependent versions of the stochastic diffusion equation. As a final example, we consider the adjoint-based approach for superconvergent recovery of linear functionals, originally developed by Pierce and Giles [29] for deterministic PDE models. In a recent study, Butler et al. [30] extended this approach to SPDE models in order to construct a posteriori error bounds. In the present paper, we consider a special class of linear output functionals depending on the solution of elliptic SPDEs. We prove that primal and adjoint-based recovery schemes are superconvergent for stochastic FE approximations using the theoretical results established in the earlier part of this paper.

The remainder of this paper is organized as follows. We set up the mathematical background and notations in Section 2. In Section 3 we provide error estimates for elliptic SPDEs (see Theorems 1 and 2). We examine in Section 4 the case of parabolic SPDEs and prove error estimates (see Theorems 3, 4, and 5) for a class of weighted temporal discretization schemes. Stability results for the weighted temporal discretization schemes are provided in Lemmas 2 and 3. In Section 5 we specify the error bound in the case of a steady-state stochastic diffusion equation (see Theorem 6) and discuss in the parabolic case the influence of the stochastic parameters of the model on the time-step restriction. In Section 6 we finally carry out an error analysis of a class of linear output functional approximations leveraging the a priori estimates derived earlier for elliptic SPDEs. Section 7 concludes the paper and outlines some directions for further work.

2. PRELIMINARIES

2.1 Notations and Definitions

To introduce our notations, we first start with the setting of elliptic SPDEs whose solution $u(\mathbf{x}\,;\boldsymbol{\xi})$ is defined on $\mathcal{D}\times\Omega$, where $\mathbf{x}\in\mathcal{D}\subset\mathbb{R}^d$ represents the spatial coordinates and $\mathcal{D}$ is an open, connected, bounded convex subset of $\mathbb{R}^d$ with polygonal boundary $\partial\mathcal{D}$. We denote the probability space by the triplet $(\Omega,\mathcal{F},\mathcal{P})$, where $\Omega\subset\mathbb{R}^q$ is the sample space, $\mathcal{F}$ is the σ-algebra associated with Ω and $\mathcal{P}$ is a probability measure. The vector $\boldsymbol{\xi}:\Omega\to\mathbb{R}^M$ represents independent real random variables with joint probability density function (pdf) $\rho(\boldsymbol{\xi})$. Throughout this paper we shall consider i.i.d uniform random variables. We denote by $\Gamma=\Gamma_1\times\cdots\times\Gamma_M$ the joint image of the random vector $\boldsymbol{\xi}$ and by $\langle\cdot\rangle$ the expectation operator with respect to ρ, that is, $\langle\cdot\rangle=\int_\Gamma\cdot\,\rho(\boldsymbol{\xi})d\boldsymbol{\xi}$.

In practice, random field discretization schemes such as KL expansions [28] are used to approximate the random fields within the SPDE model by a finite number of random variables. Hence, it can be shown that the SPDE solution $u(\mathbf{x}\,;\omega)$ with $\omega\in\Omega$ is described by a finite number of random variables, i.e., the SPDE solution $u(\mathbf{x}\,;\omega)$ is given by a deterministic parametrized PDE with solution $u(\mathbf{x}\,;\boldsymbol{\xi}(\omega))$. In the analysis that follows in this paper, we do not account for the error associated with random field discretizations; for a detailed error analysis of KL discretization, see Babuška et al. [7].

The SPDE solution u will be sought in $\mathcal{W} = \mathcal{V} \otimes \mathcal{S}$ where $\mathcal{V}$ and $\mathcal{S}$ denote Hilbert spaces of spatial and random functions, respectively. Typically, $\mathcal{V}$ will be a Sobolev space such as $H_0^1(\mathcal{D})$ or $H^1(\mathcal{D})$ and we shall consider the random function space $\mathcal{S} = L^2(\Gamma)$. When additional assumptions will be required on the stochastic regularity of the SPDE solution we shall consider $\mathcal{S} = H^k(\Gamma)$, i.e., the partial derivatives of $u(\mathbf{x}\,;\boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$, up to order k, will be assumed to be in $\mathcal{V} \otimes L^2(\Gamma)$. In other words, $||\cdot||_{\mathcal{W}}$ will denote a tensor product Sobolev norm, for example,

$$
\begin{aligned}
||u||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} &= \left(\int_\Gamma \int_{\mathcal{D}} |u(\mathbf{x}\,;\boldsymbol{\xi})|^2 \rho(\boldsymbol{\xi}) d\mathbf{x} d\boldsymbol{\xi}\right)^{1/2}, \\
||u||_{H^1(\mathcal{D})\otimes L^2(\Gamma)} &= \left(\int_\Gamma \int_{\mathcal{D}} \left(|u(\mathbf{x}\,;\boldsymbol{\xi})|^2 + |\nabla u(\mathbf{x}\,;\boldsymbol{\xi})|^2\right) \rho(\boldsymbol{\xi}) d\mathbf{x} d\boldsymbol{\xi}\right)^{1/2},
\end{aligned}
$$

and

$$
||u||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)} = \left(\int_\Gamma \int_{\mathcal{D}} |\nabla u(\mathbf{x}\,;\boldsymbol{\xi})|^2 \rho(\boldsymbol{\xi}) d\mathbf{x} d\boldsymbol{\xi}\right)^{1/2}.
$$

Note that for bounded domains $\mathcal{D}$, the norms $||\cdot||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$ and $||\cdot||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}$ are equivalent from Poincaré's inequality. More generally, the norm $||\cdot||_{L^2(\mathcal{D})\otimes H^k(\Gamma)}$ is given by

$$
||u||_{L^2(\mathcal{D})\otimes H^k(\Gamma)} = \left(\int_\Gamma \sum_{0\le|\boldsymbol{\beta}|\le k} \left|\left|D_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} u(\cdot\,;\boldsymbol{\xi})\right|\right|_{L^2(\mathcal{D})}^2 \rho(\boldsymbol{\xi}) d\boldsymbol{\xi}\right)^{1/2}, \tag{1}
$$

where the differential operator $D_{\boldsymbol{\xi}}^{\boldsymbol{\beta}}$ is defined by

$$
D_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} = \frac{\partial^{|\boldsymbol{\beta}|}}{\partial \xi_1^{\beta_1} \ldots \partial \xi_M^{\beta_M}} \tag{2}
$$

for every multi-index $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_M)$, $|\boldsymbol{\beta}| = \beta_1 + \cdots + \beta_M$.

In the case of parabolic SPDEs, we write the solution as $u(\mathbf{x}, t\,;\boldsymbol{\xi})$, where $t \in [0, T]$ denotes time ($T < +\infty$). The SPDE solution is such that $u \in L^2(0, T\,;\mathcal{W})$ and $\partial u/\partial t \in L^2(0, T\,;\mathcal{W}')$, that is,

$$
\int_0^T \left(||u(\cdot, s\,;\cdot)||_{\mathcal{W}}^2 + \left|\left|\frac{\partial u}{\partial t}(\cdot, s\,;\cdot)\right|\right|_{\mathcal{W}'}^2\right) ds < +\infty,
$$

where $\mathcal{W}'$ denotes the dual space of $\mathcal{W}$.

In the following sections, we shall outline the strong form of the elliptic and parabolic SPDE models considered in this paper along with the associated stochastic weak formulations.

2.2 Weak Form of Elliptic SPDEs

The elliptic SPDEs that we consider can be written in the general form

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} u(\mathbf{x}\,;\boldsymbol{\xi}) &= f(\mathbf{x}\,;\boldsymbol{\xi}) \quad \text{a.s. in } \mathcal{D} \times \Omega, \\
\mathcal{B}_{\boldsymbol{\xi}} u(\mathbf{x}\,;\boldsymbol{\xi}) &= g(\mathbf{x}\,;\boldsymbol{\xi}) \quad \text{a.s. on } \partial\mathcal{D} \times \Omega,
\end{aligned} \tag{3}
$$

where $\mathcal{L}_{\boldsymbol{\xi}}$ is a linear parametrized elliptic differential operator in space and $\mathcal{B}_{\boldsymbol{\xi}}$ is a parametrized operator indicating the type of boundary conditions that are imposed, namely Dirichlet, Neumann, or mixed boundary conditions. The weak formulation corresponding to (3) is given by

Find $u \in \mathcal{W}$ such that:

$$
A(u, v) = l(v), \ \forall v \in \mathcal{W}. \tag{4}
$$

We assume that the bilinear form A is continuous and elliptic with respect to the norm $||\cdot||_{\mathcal{W}}$, that is,

$$\exists \alpha_c > 0 \text{ such that } \forall u, v \in \mathcal{W},\ |A(u,v)| \leq \alpha_c\, ||u||_{\mathcal{W}} ||v||_{\mathcal{W}}, \tag{5}$$

$$\exists \alpha_e > 0 \text{ such that } \forall u \in \mathcal{W},\ A(u,u) \geq \alpha_e\, ||u||^2_{\mathcal{W}}. \tag{6}$$

We also assume that the following boundedness condition holds

$$\exists \gamma > 0 \text{ such that } |l(v)| \leq \gamma\, ||v||_{\mathcal{W}},\ \forall v \in \mathcal{W}. \tag{7}$$

The existence and uniqueness of the solution of (4) is then guaranteed by the Lax-Milgram theorem [31]. Concerning the regularity of the solution some additional problem-dependent assumptions are required. In Appendix A, the continuity and coercivity conditions (5) and (6) are proved for a class of second-order SPDEs.

2.3 Weak Form of Parabolic SPDEs

We consider parabolic SPDEs written in the general form

$$\begin{aligned} \frac{\partial u(\mathbf{x}, t\,;\boldsymbol{\xi})}{\partial t} + \mathcal{L}_{\boldsymbol{\xi}} u(\mathbf{x}, t\,;\boldsymbol{\xi}) &= f(\mathbf{x}, t\,;\boldsymbol{\xi}) && \text{a.s. in } \mathcal{D} \times [0,T] \times \Omega, \\ \mathcal{B}_{\boldsymbol{\xi}} u(\mathbf{x}, t\,;\boldsymbol{\xi}) &= g(\mathbf{x}\,;\boldsymbol{\xi}) && \text{a.s. on } \partial\mathcal{D} \times [0,T] \times \Omega, \\ u(\mathbf{x}, 0\,;\boldsymbol{\xi}) &= u_0(\mathbf{x}\,;\boldsymbol{\xi}) && \text{a.s. on } \mathcal{D} \times \Omega, \end{aligned} \tag{8}$$

where the random coefficients in $\mathcal{L}_{\boldsymbol{\xi}}$ are assumed to be independent of time. The stochastic weak problem associated with (8) is thus given by

Find $u(\cdot, t\,;\cdot) \in \mathcal{W}$ such that:

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + A(u,v) = l(v,t),\ \forall v \in \mathcal{W}, \\ u(\mathbf{x}, 0\,;\boldsymbol{\xi}) = u_0(\mathbf{x}\,;\boldsymbol{\xi}). \end{cases} \tag{9}$$

We assume that the continuity and ellipticity conditions (5) and (6) hold for the bilinear form along with the following boundedness condition:

$$\exists \gamma(t) > 0 \text{ such that } |l(v,t)| \leq \gamma(t)\, ||v||_{\mathcal{W}},\ \forall v \in \mathcal{W}. \tag{10}$$

As an example, let us consider Dirichlet boundary conditions in the SPDE model (8). Hence l is given by $l(v,t) = (f(\cdot, t\,;\cdot), v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ meaning that $\gamma(t) = ||f(\cdot, t\,;\cdot)||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$.

To guarantee the existence, uniqueness and regularity of the SPDE solution of (8), problem-dependent assumptions are needed. As an illustration, a stochastic diffusion model is discussed later in Section 5 where positivity and boundedness of the random field as well as the source term are required.

2.4 Finite-Dimensional Subspaces for Spatial and Stochastic Discretization

We shall now introduce the finite-dimensional subspaces $\mathcal{V}^h \subset \mathcal{V}$ and $\mathcal{S}^{p_{\boldsymbol{\xi}}} \subset \mathcal{S}$ used for the numerical approximation of (4) and (9). First, let $\mathscr{T}$ be a triangulation of the domain $\mathcal{D}$ consisting of a finite collection of triangles (resp. tetrahedra) T_i such that $T_i \cap T_j = \emptyset$ for $i \neq j$, $\bigcup_i T_i = \overline{\mathcal{D}}$, and such that no vertex lies in the interior of an edge (resp. a face) of another triangle (resp. tetrahedron). We consider a family of triangulations $\mathscr{T}_h$ with mesh-size $h \in [0,1[$, which are supposed to be nondegenerate, i.e., there exists a constant $\mu > 0$ such that

$$\text{diam}(B_T) \geq \mu\, \text{diam}(T),$$

for all $T \in \mathscr{T}_h$ and $h \in [0,1[$, where B_T is the largest ball contained in T, and such that

$$\max\big\{\text{diam}(T),\ T \in \mathscr{T}_h\big\} \leq h\, \text{diam}(\mathcal{D}).$$

Later in our analysis, we shall also consider quasi-uniform triangulations for which the inequality

$$\min\{\text{diam}(B_T),\ T \in \mathscr{T}_h\} \geq \mu h\,\text{diam}(\mathcal{D})$$

holds, for all $h \in [0, 1[$, $\mu > 0$. Note that quasi-uniform triangulations are nondegenerate.

The finite-dimensional subspace $\mathcal{V}^h \subset H_0^1(\mathcal{D})$ can be defined as follows

$$\mathcal{V}^h = \text{span}\{\phi_i(\mathbf{x})\}_{i=1}^n, \tag{11}$$

where $\phi_i : \mathscr{T}_h \to \mathbb{R}$ denotes a piecewise linear continuous FE basis function linked to the ith node of $\mathscr{T}_h$ and that vanishes on $\partial\mathcal{D}$. We shall generalize our analysis to smooth domains and higher-order FE approximations later on.

Using the multi-index notation $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ with $\alpha_i \in \mathbb{N}$, the M-dimensional space of polynomial chaos of degree $p_{\boldsymbol{\xi}} \in \mathbb{N}$ can be defined as [43]

$$\mathcal{S}^{p_{\boldsymbol{\xi}}} = \text{span}\left\{L_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{i=1}^{M} L_{\alpha_i}(\xi_i),\ |\boldsymbol{\alpha}| = \sum_{i=1}^{M} \alpha_i \leq p_{\boldsymbol{\xi}}\right\}, \tag{12}$$

where L_{α_i} denotes a one-dimensional Legendre polynomial of degree α_i. Note that the cardinality of $\mathcal{S}^{p_{\boldsymbol{\xi}}}$ is equal to $N_{\boldsymbol{\xi}} = [(M + p_{\boldsymbol{\xi}})!]/M!p_{\boldsymbol{\xi}}!$.

3. A PRIORI ERROR ESTIMATES FOR ELLIPTIC SPDES

Consider the solution of (4) which can be written as $u \in \mathcal{W} = \mathcal{V} \otimes \mathcal{S}$. Let $u_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$ denote the FE approximation of u given by

$$u_{h,p_{\boldsymbol{\xi}}}(\mathbf{x}\,;\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leq p_{\boldsymbol{\xi}}} u_{\boldsymbol{\alpha},h}(\mathbf{x}) L_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leq p_{\boldsymbol{\xi}}} \sum_{i=1}^{n} c^i_{\boldsymbol{\alpha},h} \phi_i(\mathbf{x}) L_{\boldsymbol{\alpha}}(\boldsymbol{\xi}). \tag{13}$$

The undetermined coefficients $c^i_{\boldsymbol{\alpha},h}$, $i = 1, 2, \dots, n$, $|\boldsymbol{\alpha}| \leq p_{\boldsymbol{\xi}}$, are computed by solving the weak form

$$A(u_{h,p_{\boldsymbol{\xi}}}, v_{h,p_{\boldsymbol{\xi}}}) = l(v_{h,p_{\boldsymbol{\xi}}}),\ \forall v_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}. \tag{14}$$

Our objective is to provide a priori error estimates for $||u - u_{h,p_{\boldsymbol{\xi}}}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$ when considering $\mathcal{V} = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$ and $\mathcal{S} = H^k(\Gamma)$, i.e., under the assumption that the SPDE solution satisfies some spatial and stochastic regularity conditions; see Theorem 1 for the general case. Under additional problem-dependent assumptions, sharper error bounds can be obtained in the norm $||\cdot||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ using a duality argument; see Theorem 2. It is worth mentioning here that a priori error estimates can also be obtained for Galerkin approximations of stochastic diffusion models under the assumption that the SPDE solution satisfies some analyticity conditions. More specifically, Nobile, Tempone, and co-workers provide exponential convergence rates [9] when the SPDE solution obeys polydisc analyticity assumptions in the complex plane, using an approximation result for holomorphic functions by Bagby et al. [32].

Using (4)–(6) and (14) we obtain the following inequality from Céa's lemma:

$$||u - u_{h,p_{\boldsymbol{\xi}}}||_{\mathcal{V}\otimes L^2(\Gamma)} \leq \frac{\alpha_c}{\alpha_e} ||u - v_{h,p_{\boldsymbol{\xi}}}||_{\mathcal{V}\otimes L^2(\Gamma)},\ \forall v_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}. \tag{15}$$

In order to proceed further, we need to estimate an upper bound for $||u - v_{h,p_{\boldsymbol{\xi}}}||_{\mathcal{V}\otimes L^2(\Gamma)}$ for any interpolant $v_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$. Consider the splitting $u - v_{h,p_{\boldsymbol{\xi}}} = u - v_{p_{\boldsymbol{\xi}}} + v_{p_{\boldsymbol{\xi}}} - v_{h,p_{\boldsymbol{\xi}}}$ where $v_{p_{\boldsymbol{\xi}}}$ is the projection of u onto $\mathcal{V} \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$, that is,

$$v_{p_{\boldsymbol{\xi}}}(\mathbf{x}\,;\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leq p_{\boldsymbol{\xi}}} v_{\boldsymbol{\alpha}}(\mathbf{x}) L_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{16}$$

with

$$v_{\boldsymbol{\alpha}}(\mathbf{x}) = \frac{(u(\mathbf{x}\,;\cdot), L_{\boldsymbol{\alpha}})_{L^2(\Gamma)}}{||L_{\boldsymbol{\alpha}}||^2_{L^2(\Gamma)}}. \tag{17}$$

Using the orthogonality of the PC basis $L_{\boldsymbol{\alpha}}$ in $L^2(\Gamma)$, we get

$$||u - v_{h,p_{\boldsymbol{\xi}}}||^2_{\mathcal{V}\otimes L^2(\Gamma)} = ||u - v_{p_{\boldsymbol{\xi}}}||^2_{\mathcal{V}\otimes L^2(\Gamma)} + ||v_{p_{\boldsymbol{\xi}}} - v_{h,p_{\boldsymbol{\xi}}}||^2_{\mathcal{V}\otimes L^2(\Gamma)}. \tag{18}$$

We first focus on $p_{\boldsymbol{\xi}}$*-error estimation* (see Lemma 1) corresponding to the stochastic discretization and then provide the full $p_{\boldsymbol{\xi}}$-h *error estimation* (see Theorem 1) which accounts for both the spatial and stochastic discretization parameters.

Lemma 1. Let $u \in \mathcal{V} \otimes H^k(\Gamma)$ and $v_{p_{\boldsymbol{\xi}}} \in \mathcal{V} \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$ defined by (16) with $\mathcal{S}^{p_{\boldsymbol{\xi}}}$ given by (12). Then the following inequality holds:

$$||u - v_{p_{\boldsymbol{\xi}}}||_{\mathcal{V}\otimes L^2(\Gamma)} \leq C(M)\, p_{\boldsymbol{\xi}}^{-k}\, ||u||_{\mathcal{V}\otimes H^k(\Gamma)}, \tag{19}$$

where C is a constant independent of $p_{\boldsymbol{\xi}}$ that grows linearly with M.

Proof. This result is proved in Appendix B. □

We are now in a position to derive error estimates for $||u - u_{h,p_{\boldsymbol{\xi}}}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$ when considering $\mathcal{V} = (H^1_0(\mathcal{D}) \cap H^2(\mathcal{D}))$ and $\mathcal{S} = H^k(\Gamma)$.

Theorem 1. Let $u \in (H^1_0(\mathcal{D}) \cap H^2(\mathcal{D})) \otimes H^k(\Gamma)$ be the solution of the weak formulation (4) and $u_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$ denote the solution of (14). The bilinear form A is assumed to be α_c-continuous and α_e-elliptic with respect to the norm $||\cdot||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$. Then the following error estimate holds:

$$||u - u_{h,p_{\boldsymbol{\xi}}}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c}{\alpha_e}\left(C(M)\, p_{\boldsymbol{\xi}}^{-k}\, ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\right), \tag{20}$$

where C and C^* are constants independent of $p_{\boldsymbol{\xi}}$ and h.

Proof. Coming back to the inequality (15) given by Céa's lemma, and using the splitting (18), we have

$$||u - u_{h,p_{\boldsymbol{\xi}}}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c^2}{\alpha_e^2}\left(||u - v_{p_{\boldsymbol{\xi}}}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} + ||v_{p_{\boldsymbol{\xi}}} - v_{h,p_{\boldsymbol{\xi}}}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)}\right).$$

Applying the $p_{\boldsymbol{\xi}}$-error estimate (19) yields

$$||u - u_{h,p_{\boldsymbol{\xi}}}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c^2}{\alpha_e^2}\left(C(M)^2 p_{\boldsymbol{\xi}}^{-2k}||u||^2_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + ||v_{p_{\boldsymbol{\xi}}} - v_{h,p_{\boldsymbol{\xi}}}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)}\right). \tag{21}$$

To estimate the second term in the right-hand side of (21), we expand $v_{p_{\boldsymbol{\xi}}}$ and $v_{h,p_{\boldsymbol{\xi}}}$ in an orthogonal Legendre PC basis, which gives

$$||v_{p_{\boldsymbol{\xi}}} - v_{h,p_{\boldsymbol{\xi}}}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} = \sum_{|\boldsymbol{\alpha}|\leq p_{\boldsymbol{\xi}}} ||v_{\boldsymbol{\alpha}} - v_{\boldsymbol{\alpha},h}||^2_{H^1(\mathcal{D})}\, ||L_{\boldsymbol{\alpha}}||^2_{L^2(\Gamma)}.$$

Using the classical interpolation error estimate for piecewise linear basis functions,

$$||v_{\boldsymbol{\alpha}} - v_{\boldsymbol{\alpha},h}||_{H^1(\mathcal{D})} \leq C^* h\, ||v_{\boldsymbol{\alpha}}||_{H^2(\mathcal{D})},$$

where C^* is a constant independent of h, we get

$$||v_{p_{\boldsymbol{\xi}}} - v_{h,p_{\boldsymbol{\xi}}}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \quad \leq \quad C^{*2}h^2 \sum_{|\boldsymbol{\alpha}|\leq p_{\boldsymbol{\xi}}} ||v_{\boldsymbol{\alpha}}||^2_{H^2(\mathcal{D})}||L_{\boldsymbol{\alpha}}||^2_{L^2(\Gamma)} \leq C^{*2}h^2||u||^2_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \tag{22}$$

since $u(\mathbf{x}\,;\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha}} v_{\boldsymbol{\alpha}}(\mathbf{x}) L_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$. Combining (21) and (22) leads to (20) and concludes the proof. □

In the a priori error estimate (20), the number of random variables M appears in the constant C and in the norms $||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)}$ and $||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}$ through $\Gamma = \Gamma_1 \times \cdots \times \Gamma_M$. If we assume that the norm $||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)}$ of the analytical solution can be bounded independently of M, then it can be seen from (20) that the stochastic discretization error (without taking into account the error associated with random field discretization) tends to zero when $p_{\boldsymbol{\xi}} \to +\infty$, for any fixed value of M. From a practical point of view, it can be more convenient to consider the stochastic convergence rate in terms of the total number of PC basis functions ($N_{\boldsymbol{\xi}}$) and the total number of random variables (M) instead of the PC order ($p_{\boldsymbol{\xi}}$) [9]. From the definition of $N_{\boldsymbol{\xi}} = N_{\boldsymbol{\xi}}(M, p_{\boldsymbol{\xi}})$, we can estimate $p_{\boldsymbol{\xi}}^{-k}$ as a function of M and $N_{\boldsymbol{\xi}}$ and rewrite the error estimate as follows.

Corollary 1. Under the same assumptions as in Theorem 1, the error estimate (20) can be written as

$$||u - u_{h,p_{\boldsymbol{\xi}}}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c}{\alpha_e}\left(C(M)\left(\frac{1+\log(M)}{\log(N_{\boldsymbol{\xi}})}\right)^k ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\right). \tag{23}$$

Proof. The number of PC basis functions $N_{\boldsymbol{\xi}}$ can be estimated as follows (see [13] and [9]):

$$N_{\boldsymbol{\xi}} = \frac{(M+p_{\boldsymbol{\xi}})!}{M!p_{\boldsymbol{\xi}}!} = \prod_{j=1}^{M}\left(1+\frac{p_{\boldsymbol{\xi}}}{j}\right) \leq \prod_{j=1}^{M}\exp\left(\frac{p_{\boldsymbol{\xi}}}{j}\right) = \exp\left(p_{\boldsymbol{\xi}}\sum_{j=1}^{M}\frac{1}{j}\right) \leq \exp\big(p_{\boldsymbol{\xi}}(1+\log(M))\big)$$

leading to the inequality

$$\frac{1}{p_{\boldsymbol{\xi}}} \leq \frac{1+\log(M)}{\log(N_{\boldsymbol{\xi}})}. \tag{24}$$

Substituting (24) in (20) gives (23). □

Corollary 1 indicates that if the norm $||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)}$ can be bounded independently of M, then the stochastic error tends to zero when $M \to +\infty$ and $p_{\boldsymbol{\xi}} \to +\infty$. It is to be noted that the boundedness of $||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)}$ with respect to M is problem dependent.

We shall now derive a sharper error estimate in the norm $||\cdot||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ using duality arguments when considering elliptic SPDEs with Dirichlet boundary conditions. In this case, the weak form writes as
Find $u \in \mathcal{V}\otimes\mathcal{S}$ such that:

$$A(u,v) = (f,v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall v \in \mathcal{V}\otimes\mathcal{S}, \tag{25}$$

with $\mathcal{V} = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$ and $\mathcal{S} = H^k(\Gamma)$, while the adjoint (or dual) variational problem associated with (25) is given by
Find $w \in \mathcal{V}\otimes\mathcal{S}$ such that:

$$A(v,w) = (g,v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall v \in \mathcal{V}\otimes\mathcal{S}. \tag{26}$$

We shall assume that the adjoint solution satisfies the regularity conditions

$$\begin{aligned} ||w||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} &\leq C_r||g||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}, \\ ||w||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} &\leq \widetilde{C}_r||g||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}, \end{aligned} \tag{27}$$

for all $g \in L^2(\mathcal{D})\otimes L^2(\Gamma)$, where C_r and $\widetilde{C}_r$ are constants independent of w.

Theorem 2. Let u denote the solution of (25) and let $u_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$ be its FE-based approximate solution. Under the regularity conditions (27), the following error estimate holds:

$$\begin{aligned} ||u - u_{h,p_{\boldsymbol{\xi}}}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c^2}{\alpha_e}\Big(&\big(A_1(M)||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + A_2(M)||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\big)\, p_{\boldsymbol{\xi}}^{-2k} \\ &+ \big(B_1(M)||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + B_2(M)||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\big)\, h^2\Big), \end{aligned} \tag{28}$$

where A_1, A_2, B_1, and B_2 are constants independent of $p_{\boldsymbol{\xi}}$ and h.

Proof. Let w be the solution of (26) with $g = u - u_{h,p_\xi}$. We have

$$||u - u_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = (u - u_{h,p_\xi}, u - u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = A(u - u_{h,p_\xi}, w) = A(u - u_{h,p_\xi}, w - w_{h,p_\xi}),$$

since $A(u - u_{h,p_\xi}, w_{h,p_\xi}) = 0$ for all $w_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$. Thus, we get:

$$||u - u_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \alpha_c ||u - u_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)} ||w - w_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}.$$

An upper bound for the term $||u - u_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$ can be obtained using (20). Concerning the second term, we can use the error estimate that holds for interpolants (using a tensor product of linear FE basis functions and Legendre PC basis functions) in $\mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$, that is,

$$\inf_{v_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}} ||u - v_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq C(M)\, p_\xi^{-k}\, ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}.$$

Hence, it follows that

$$\begin{aligned} ||u - u_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq\ & \frac{\alpha_c^2}{\alpha_e}\Big(C(M)\, p_\xi^{-k}\, ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \Big) \\ & \times \Big(C(M)\, p_\xi^{-k}\, ||w||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||w||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \Big). \end{aligned}$$

From (27), we have

$$||w||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \leq C_r ||u - u_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)},$$

$$||w||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} \leq \widetilde{C}_r ||u - u_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)},$$

which yields

$$\begin{aligned} & ||u - u_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \\ & \leq \frac{\alpha_c^2}{\alpha_e}\Big(C(M)\, p_\xi^{-k}\, ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \Big) \times \Big(C(M)\widetilde{C}_r p_\xi^{-k} + C^* C_r h \Big) \\ & = \frac{\alpha_c^2}{\alpha_e}\Big(C(M)^2 \widetilde{C}_r ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)}\, p_\xi^{-2k} + C^{*2} C_r ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\, h^2 \\ & + C(M) C^* \Big(C_r ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + \widetilde{C}_r ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \Big)\, h p_\xi^{-k} \Big). \end{aligned}$$

Using the inequality $hp_\xi^{-k} \leq (1/2)(h^2 + p_\xi^{-2k})$ and reordering terms, we obtain the final error estimate (28) with $A_1(M) = C(M)^2 \widetilde{C}_r + [C(M)C^* C_r/2]$, $A_2(M) = C(M)C^* \widetilde{C}_r/2$, $B_1(M) = C(M)C^* C_r/2$, and $B_2(M) = C^{*2} C_r + [C(M)C^* \widetilde{C}_r/2]$. This concludes the proof. □

4. A PRIORI ERROR ESTIMATES FOR PARABOLIC SPDES

4.1 Time-Stepping Scheme

We now focus on numerical solution of the weak form (9) where Dirichlet boundary conditions are considered in the parabolic SPDE model (8). We consider a θ-weighted temporal discretization scheme with $\theta \in [0, 1]$ which results in the following weak problem:

Find $u^m_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$, $0 \le m \le N_t$, such that:

$$\begin{cases} \left(\dfrac{u^{m+1}_{h,p_\xi} - u^m_{h,p_\xi}}{\Delta t}, v_{h,p_\xi}\right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + A(u^{m+\theta}_{h,p_\xi}, v_{h,p_\xi}) = (f^{m+\theta}_{h,p_\xi}, v_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)}, \\ (u^0_{h,p_\xi} - u_0, v_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = 0, \end{cases} \tag{29}$$

for all $v_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$, where u^m_{h,p_ξ} and f^m_{h,p_ξ} denote the approximate solution and source term computed at time $t^m = m\Delta t$, for $0 \le m \le N_t$ and $\Delta t = T/N_t$. In (29) $u^{m+\theta}_{h,p_\xi}$ and $f^{m+\theta}_{h,p_\xi}$ are defined as

$$u^{m+\theta}_{h,p_\xi} = \theta\, u^{m+1}_{h,p_\xi} + (1-\theta)\, u^m_{h,p_\xi}, \tag{30}$$

and

$$f^{m+\theta}_{h,p_\xi} = \theta\, f^{m+1}_{h,p_\xi} + (1-\theta)\, f^m_{h,p_\xi}. \tag{31}$$

The cases $\theta = 0$, $\theta = 1/2$, and $\theta = 1$ correspond to the forward Euler, Crank-Nicolson, and backward Euler schemes, respectively. Our aim is to estimate an upper bound for the error metric

$$\max_{m=1,\dots,N_t} ||u(\cdot, t^m\,; \cdot) - u^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}, \tag{32}$$

where u and u^m_{h,p_ξ} denote the solutions of (9) and (29), respectively.

4.2 Stability Analysis

In this section, we prove stability results for the temporal discretization scheme (29) using ideas from the analysis of Süli [33]. We first show that (29) is unconditionally stable for $\theta \in [1/2, 1]$ (see Lemma 2) and then we prove that (29) is conditionally stable for $\theta \in [0, 1/2[$ under some restrictions (see Lemma 3).

Lemma 2. Let $u^m_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$ be the solution of (29) with $\theta \in [1/2, 1]$, where A is α_c-continuous and α_e-elliptic on $\mathcal{V} \otimes \mathcal{S}$ with respect to the norm $||\cdot||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$. The spatial triangulation of $\mathcal{D} \subset \mathbb{R}^d$ is nondegenerate. Then the following inequality holds:

$$\max_{k=1,\dots N_t} ||u^k_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \le ||u^0_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{\Delta t}{\alpha_e} \sum_{m=0}^{N_t-1} ||f^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}, \tag{33}$$

where $f^{m+\theta}_{h,p_\xi}$ is given by (31).

Proof. This result is proved in Appendix C.1. □

Lemma 3. Consider the same assumptions as in Lemma 2 with a quasi-uniform spatial discretization of $\mathcal{D}$. For $\theta \in [0, 1/2[$, the following stability condition

$$\max_{k=1,\dots N_t} ||u^k_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \le ||u^0_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \Delta t\, c_\epsilon \sum_{m=0}^{N_t-1} ||f^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}, \tag{34}$$

holds under the assumptions

$$\frac{\Delta t}{h^2} \le \frac{2\alpha_e(C^2_{\mathcal{D}}+1) - 4\epsilon^2 C^2_{\mathcal{D}}}{(C^2_{\mathcal{D}}+1)(1-2\theta)\alpha_c^2 (C_i^*)^2 (1+\epsilon)},\ 0 < \epsilon \le \left(\frac{\alpha_e(1+C^2_{\mathcal{D}})}{2C^2_{\mathcal{D}}}\right)^{1/2}, \tag{35}$$

where C_i^* is defined by (C.4) and c_ϵ is given by $c_\epsilon = (1-2\theta)\,(1+(1/\epsilon))\,\Delta t + (1/4\epsilon^2)$.

Proof. A proof of this result can be found in Appendix C.2. □

We shall discuss the stability analysis results in the context of the stochastic diffusion equation later in Section 5.3. In particular, we shall discuss the role played by the correlation length and standard deviation of the random field on the time-step restriction.

4.3 A Priori Error Estimates

Theorem 3. Let $u \in L^2(0,T\,;\mathcal{W})$ be the solution of the weak form (9) with Dirichlet boundary conditions, where $\mathcal{W} = (H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})) \otimes H^k(\Gamma)$ and $\partial u/\partial t \in L^2(0,T\,;\mathcal{W}')$. We assume that the source term f and its temporal derivatives are smooth enough and that the initial value u_0 belongs at least to $H^2(\mathcal{D}) \otimes H^k(\Gamma)$. The bilinear form A is assumed to be α_c-continuous and α_e-elliptic with respect to the norm $||\cdot||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$. Let $u^m_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$ denote the solution of the θ-scheme (29) with $\theta \in [1/2, 1]$, where the finite-dimensional spaces $\mathcal{V}^h$ and $\mathcal{S}^{p_\xi}$ are given by (11) and (12), respectively. The spatial triangulation is supposed to be nondegenerate. Then the following error estimate holds:

$$\max_{m=1,\ldots,N_t} ||u(\cdot,t^m\,;\cdot) - u^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq C_1\, h + C_2\, \Delta t + C_3\, p_\xi^{-k}, \tag{36}$$

where the constants $C_1, C_2, C_3 > 0$ are independent of h and Δt, and only depend on the analytical solution as follows:

$$C_1 = \frac{C^*\alpha_c}{\alpha_e}\left(\max_{m=1,2,\ldots,N_t} ||u(\cdot,t^m\,,\cdot)||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} + ||u_0||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{2}{\sqrt{\alpha_e}} K_1(u,T)\right),$$

$$C_2 = \sqrt{\frac{2}{\alpha_e}}\, K_2(u,T),$$

$$C_3 = \frac{C(M)\alpha_c}{\alpha_e}\left(\max_{m=1,2,\ldots,N_t} ||u(\cdot,t^m\,,\cdot)||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + ||u_0||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + \frac{2}{\sqrt{\alpha_e}} K_3(u,T)\right),$$

The constants $C(M)$ and C^* are independent of p_ξ and h, and the constants K_1, K_2, and K_3 are given by

$$K_1(u,T) = \left(\int_0^T \left|\left|\frac{\partial u}{\partial t}(\cdot,s\,;\cdot)\right|\right|^2_{H^2(\mathcal{D})\otimes L^2(\Gamma)} ds\right)^{1/2}, \tag{37}$$

$$K_2(u,T) = \left(\int_0^T \left|\left|\frac{\partial^2 u}{\partial t^2}(\cdot,s\,;\cdot)\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ds\right)^{1/2}, \tag{38}$$

$$K_3(u,T) = \left(\int_0^T \left|\left|\frac{\partial u}{\partial t}(\cdot,s\,;\cdot)\right|\right|^2_{H^1(\mathcal{D})\otimes H^k(\Gamma)} ds\right)^{1/2}. \tag{39}$$

Proof. We first split the approximation error as

$$e^m_{h,p_\xi} = u(\cdot,t^m\,;\cdot) - u^m_{h,p_\xi} = \underbrace{u(\cdot,t^m\,;\cdot) - Pu(\cdot,t^m\,;\cdot)}_{\eta^m_{h,p_\xi}} + \underbrace{Pu(\cdot,t^m\,;\cdot) - u^m_{h,p_\xi}}_{\xi^m_{h,p_\xi}}, \tag{40}$$

where $P : \mathcal{V} \otimes \mathcal{S} \to \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$ is the elliptic projection defined by the Galerkin conditions:

$$A(Pu(\cdot,t\,;\cdot), v_{h,p_\xi}) = A(u(\cdot,t\,;\cdot), v_{h,p_\xi}),\ \forall v_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}.$$

The existence and uniqueness of $Pu(\cdot,t\,;\cdot)$ is a consequence of the Lax-Milgram theorem (see [31]). Hence we need to estimate

$$\max_{m=1,\ldots,N_t} ||e^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \max_{m=1,\ldots,N_t} ||\eta^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \max_{m=1,\ldots,N_t} ||\xi^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \tag{41}$$

Since we have

$$A(\eta^m_{h,p_\xi}, v_{h,p_\xi}) = 0,\ \forall v_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi},$$

we can use the error estimate (20) that we proved earlier in Theorem 1 for elliptic SPDEs,

$$||\eta^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq ||\eta^m_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$$
$$\leq \frac{\alpha_c}{\alpha_e}\Big(C(M)\, p_\xi^{-k}\, ||u(\cdot,t^m\,;\cdot)||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^*\, h\, ||u(\cdot,t^m\,;\cdot)||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\Big). \quad (42)$$

We focus now on estimating an upper bound for $||\xi^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$. From (40) and (30) we substitute

$$u^m_{h,p_\xi} = u(\cdot,t^m\,;\cdot) - (\eta^m_{h,p_\xi} + \xi^m_{h,p_\xi})$$

and

$$u^{m+\theta}_{h,p_\xi} = \theta\left(u(\cdot,t^{m+1}\,;\cdot) - (\eta^{m+1}_{h,p_\xi} + \xi^{m+1}_{h,p_\xi})\right) + (1-\theta)\left(u(\cdot,t^m\,;\cdot) - (\eta^m_{h,p_\xi} + \xi^m_{h,p_\xi})\right)$$

into the θ-scheme (29). Using (31), we note that ξ^m_{h,p_ξ} are the solution of

$$\left(\frac{\xi^{m+1}_{h,p_\xi} - \xi^m_{h,p_\xi}}{\Delta t}, v_{h,p_\xi}\right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + A(\xi^{m+\theta}_{h,p_\xi}, v_{h,p_\xi}) = (\varphi^{m+\theta}_{h,p_\xi}, v_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \quad (43)$$

with

$$\varphi^{m+\theta}_{h,p_\xi} = \frac{u(\cdot,t^{m+1}\,;\cdot) - u(\cdot,t^m\,;\cdot)}{\Delta t} - \frac{\partial u}{\partial t}(\cdot,t^{m+\theta}\,;\cdot) - \frac{\eta^{m+1}_{h,p_\xi} - \eta^m_{h,p_\xi}}{\Delta t}$$

and

$$\frac{\partial u}{\partial t}(\cdot,t^{m+\theta}\,;\cdot) = \theta\,\frac{\partial u}{\partial t}(\cdot,t^{m+1}\,;\cdot) + (1-\theta)\,\frac{\partial u}{\partial t}(\cdot,t^m\,;\cdot).$$

We then apply the stability result (33) to (43), which yields

$$\max_{m=1,\ldots,N_t} ||\xi^m_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq ||\xi^0_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{\Delta t}{\alpha_e}\sum_{m=0}^{N_t-1} ||\varphi^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \quad (44)$$

with

$$||\varphi^{m+\theta}_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \underbrace{\left|\left|\frac{u(\cdot,t^{m+1}\,;\cdot) - u(\cdot,t^m\,;\cdot)}{\Delta t} - \frac{\partial u}{\partial t}(\cdot,t^{m+\theta}\,;\cdot)\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)}}_{(I)}$$
$$+ \underbrace{\left|\left|\frac{\eta^{m+1}_{h,p_\xi} - \eta^m_{h,p_\xi}}{\Delta t}\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)}}_{(II)}. \quad (45)$$

We first consider the estimation of (I). Noting that

$$\frac{u(\cdot,t^{m+1}\,;\cdot) - u(\cdot,t^m\,;\cdot)}{\Delta t} - \frac{\partial u}{\partial t}(\cdot,t^{m+\theta}\,;\cdot) = \theta\left(\frac{u(\cdot,t^{m+1}\,;\cdot) - u(\cdot,t^m\,;\cdot)}{\Delta t} - \frac{\partial u}{\partial t}(\cdot,t^{m+1}\,;\cdot)\right)$$
$$+\ (1-\theta)\left(\frac{u(\cdot,t^{m+1}\,;\cdot) - u(\cdot,t^m\,;\cdot)}{\Delta t} - \frac{\partial u}{\partial t}(\cdot,t^m\,;\cdot)\right)$$

and using Taylor's formulas with integral remainders

$$u(\cdot,t^{m+1}\,;\cdot) = u(\cdot,t^m\,;\cdot) + \Delta t\,\frac{\partial u}{\partial t}(\cdot,t^m\,;\cdot) + \int_{t^m}^{t^{m+1}} (t^{m+1}-s)\frac{\partial^2 u}{\partial t^2}(\cdot,s\,;\cdot)ds$$

$$u(\cdot,t^m\,;\cdot) = u(\cdot,t^{m+1}\,;\cdot) - \Delta t\,\frac{\partial u}{\partial t}(\cdot,t^{m+1}\,;\cdot) + \int_{t^{m+1}}^{t^m} (t^m-s)\frac{\partial^2 u}{\partial t^2}(\cdot,s\,;\cdot)ds,$$

we have

$$(I) \leq \frac{\theta}{\Delta t}\left|\left|\int_{t^m}^{t^{m+1}} (t^m - s)\,\frac{\partial^2 u}{\partial t^2}(\cdot, s\,;\cdot)ds\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{1-\theta}{\Delta t}\left|\left|\int_{t^m}^{t^{m+1}} (t^{m+1} - s)\,\frac{\partial^2 u}{\partial t^2}(\cdot, s\,;\cdot)ds\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)}.$$

Applying the Minkowsky inequality, we get

$$(I) \leq \frac{\theta}{\Delta t}\int_{t^m}^{t^{m+1}} |t^m - s|\left|\left|\frac{\partial^2 u}{\partial t^2}(\cdot, s\,;\cdot)\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ds + \frac{1-\theta}{\Delta t}\int_{t^m}^{t^{m+1}} |t^{m+1} - s|\left|\left|\frac{\partial^2 u}{\partial t^2}(\cdot, s\,;\cdot)\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ds.$$

Since $|t^m - s| \leq \Delta t$ and $|t^{m+1} - s| \leq \Delta t$, it follows that

$$(I) \leq \int_{t^m}^{t^{m+1}} \left|\left|\frac{\partial^2 u}{\partial t^2}(\cdot, s\,;\cdot)\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ds \leq \sqrt{\Delta t}\left(\int_{t^m}^{t^{m+1}} \left|\left|\frac{\partial^2 u}{\partial t^2}(\cdot, s\,;\cdot)\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ds\right)^{1/2} = \sqrt{\Delta t}\,\alpha_m(u) \quad (46)$$

using the Cauchy-Schwarz inequality.

Concerning the second term (II), we use the fact that

$$A\left(\frac{\eta^{m+1}_{h,p_\xi} - \eta^{m}_{h,p_\xi}}{\Delta t}, v_{h,p_\xi}\right) = 0,\ \forall v_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}.$$

Using the error estimate (20) again leads to

$$\begin{aligned}(II) \leq \frac{\alpha_c}{\alpha_e}\Bigg(& C(M)\,p_\xi^{-k}\left|\left|\frac{u(\cdot, t^{m+1}\,;\cdot) - u(\cdot, t^m\,;\cdot)}{\Delta t}\right|\right|_{H^1(\mathcal{D})\otimes H^k(\Gamma)} \\ &+ C^*\,h\left|\left|\frac{u(\cdot, t^{m+1}\,;\cdot) - u(\cdot, t^m\,;\cdot)}{\Delta t}\right|\right|_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\Bigg). \end{aligned} \quad (47)$$

We have

$$\left|\left|\frac{u(\cdot, t^{m+1}\,;\cdot) - u(\cdot, t^m\,;\cdot)}{\Delta t}\right|\right|_{H^1(\mathcal{D})\otimes H^k(\Gamma)} = \frac{1}{\Delta t}\left|\left|\int_{t^m}^{t^{m+1}} \frac{\partial u}{\partial t}(\cdot, s\,;\cdot)ds\right|\right|_{H^1(\mathcal{D})\otimes H^k(\Gamma)} = \frac{1}{\Delta t}||w||_{H^1(\mathcal{D})\otimes H^k(\Gamma)},$$

and

$$||w||^2_{H^1(\mathcal{D})\otimes H^k(\Gamma)} = \sum_{0\leq|\beta|\leq k}\int_\Gamma ||D^\beta_\xi w(\cdot\,;\xi)||^2_{H^1(\mathcal{D})}\rho(\xi)d\xi,$$

where D^β_ξ is given by (2). Applying Minkowsky and Cauchy-Schwarz's inequalities, we get

$$||D^\beta_\xi w(\cdot\,;\xi)||_{H^1(\mathcal{D})} \leq \sqrt{\Delta t}\left(\int_{t^m}^{t^{m+1}} \left|\left|D^\beta_\xi \frac{\partial u}{\partial t}(\cdot, s\,;\cdot)\right|\right|^2_{H^1(\mathcal{D})} ds\right)^{1/2},$$

which yields

$$\begin{aligned}\left|\left|\frac{u(\cdot, t^{m+1}\,;\cdot) - u(\cdot, t^m\,;\cdot)}{\Delta t}\right|\right|_{H^1(\mathcal{D})\otimes H^k(\Gamma)} &\leq \frac{1}{\sqrt{\Delta t}}\left(\sum_{0\leq|\beta|\leq k}\int_\Gamma\int_{t^m}^{t^{m+1}} \left|\left|D^\beta_\xi \frac{\partial u}{\partial t}(\cdot, s\,;\cdot)\right|\right|^2_{H^1(\mathcal{D})} ds\rho(\xi)d\xi\right)^{1/2} \\ &= \frac{1}{\sqrt{\Delta t}}\,\beta_m(u). \end{aligned} \quad (48)$$

Similarly, we get the following inequality

$$\left|\left|\frac{u(\cdot,t^{m+1};\cdot)-u(\cdot,t^{m};\cdot)}{\Delta t}\right|\right|_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{1}{\sqrt{\Delta t}}\left(\int_\Gamma\int_{t^m}^{t^{m+1}}\left|\left|\frac{\partial u}{\partial t}(\cdot,s;\cdot)\right|\right|^2_{H^2(\mathcal{D})} ds\rho(\boldsymbol{\xi})d\boldsymbol{\xi}\right)^{1/2}$$
$$= \frac{1}{\sqrt{\Delta t}}\,\gamma_m(u). \tag{49}$$

From (47)–(49), it follows that

$$(II) \leq \frac{1}{\sqrt{\Delta t}}\frac{\alpha_c}{\alpha_e}\left(C(M)\,\beta_m(u)\,p_{\boldsymbol{\xi}}^{-k} + C^*\,\gamma_m(u)\,h\right). \tag{50}$$

To complete the estimation in (44), we invoke the Galerkin orthogonality condition related to the initial value, which gives

$$(u^0_{h,p_{\boldsymbol{\xi}}} - u_0, v_{h,p_{\boldsymbol{\xi}}})_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall v_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h\otimes\mathcal{S}^{p_{\boldsymbol{\xi}}}. \tag{51}$$

Noting that $u^0_{h,p_{\boldsymbol{\xi}}} - u_0 = -(\eta^0_{h,p_{\boldsymbol{\xi}}} + \xi^0_{h,p_{\boldsymbol{\xi}}})$, and taking $v_{h,p_{\boldsymbol{\xi}}} = \xi^0_{h,p_{\boldsymbol{\xi}}}$ in (51), we get

$$||\xi^0_{h,p_{\boldsymbol{\xi}}}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq ||\eta^0_{h,p_{\boldsymbol{\xi}}}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c}{\alpha_e}\left(C(M)\,p_{\boldsymbol{\xi}}^{-k}\,||u_0||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^*\,h\,||u_0||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\right). \tag{52}$$

Combining (41), (42), (44), (45), (46), (50), (52) and reordering terms, we have

$$\begin{aligned}\max_{m=1,2,\ldots N_t}||e^m_{h,p_{\boldsymbol{\xi}}}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq{}& \frac{C^*\alpha_c}{\alpha_e}h\Bigg[\max_{m=1,2,\ldots,N_t}||u(\cdot,t^m\ ,\cdot)||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} + ||u_0||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\\ &+ \frac{2}{\sqrt{\alpha_e}}\left(\sum_{m=0}^{N_t-1}(\gamma_m(u))^2\right)^{1/2}\Bigg] + \sqrt{\frac{2}{\alpha_e}}\Delta t\left(\sum_{m=0}^{N_t-1}(\alpha_m(u))^2\right)^{1/2}\\ &+ \frac{C(M)\alpha_c}{\alpha_e}p_{\boldsymbol{\xi}}^{-k}\Bigg[\max_{m=1,2,\ldots,N_t}||u(\cdot,t^m\ ,\cdot)||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + ||u_0||_{H^1(\mathcal{D})\otimes H^k(\Gamma)}\\ &+ \frac{2}{\sqrt{\alpha_e}}\left(\sum_{m=0}^{N_t-1}(\beta_m(u))^2\right)^{1/2}\Bigg].\end{aligned}$$

In the preceding inequality, the upper bound still depends on Δt through the summations over N_t terms, $N_t = T/\Delta t$. To proceed further and obtain an upper bound independent of Δt, we use the following relation:

$$\begin{aligned}\sum_{m=0}^{N_t-1}(\gamma_m(u))^2 &= \sum_{m=0}^{N_t-1}\int_\Gamma\int_{t^m}^{t^{m+1}}\left|\left|\frac{\partial u}{\partial t}(\cdot,s;\cdot)\right|\right|^2_{H^2(\mathcal{D})} ds\,\rho(\boldsymbol{\xi})d\boldsymbol{\xi}\\ &= \int_0^T\left|\left|\frac{\partial u}{\partial t}(\cdot,s;\cdot)\right|\right|^2_{H^2(\mathcal{D})\otimes L^2(\Gamma)} ds = (K_1(u,T))^2,\end{aligned}$$

and similarly

$$\sum_{m=0}^{N_t-1}(\alpha_m(u))^2 = \int_0^T\left|\left|\frac{\partial^2 u}{\partial t^2}(\cdot,s;\cdot)\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ds = (K_2(u,T))^2,$$

$$\sum_{m=0}^{N_t-1}(\beta_m(u))^2 = \int_0^T\left|\left|\frac{\partial u}{\partial t}(\cdot,s;\cdot)\right|\right|^2_{H^1(\mathcal{D})\otimes H^k(\Gamma)} ds = (K_3(u,T))^2.$$

The assumption that the source term f and its temporal derivatives are smooth enough and that the initial value u_0 belongs at least to $H^2(\mathcal{D}) \otimes H^k(\Gamma)$ ensures that $||u_0||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}$ and $||u_0||_{H^1(\mathcal{D})\otimes H^k(\Gamma)}$ are finite quantities and that $\partial u/\partial t \in L^2(0,T\,;H^2(\mathcal{D}) \otimes H^k(\Gamma))$ and $\partial^2 u/\partial t^2 \in L^2(0,T\,;L^2(\mathcal{D}) \otimes L^2(\Gamma))$. This means that the constants K_1, K_2, and K_3 are well defined. This leads to the final a priori error estimate (36) and concludes the proof. □

Theorem 4. Under the same assumptions as in Theorem 3, assume in addition that the dual solution of the steady-state model satisfies the regularity conditions (27). Then the following a priori error estimate holds:

$$\max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot) - u^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq C_1\, h^2 + C_2\, \Delta t + C_3\, p_\xi^{-2k} \tag{53}$$

with

$$\begin{aligned}
C_1 &= \frac{\alpha_c^2}{\alpha_e}\Bigg(B_1 \max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot)||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + B_2 \max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot)||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \\
&\quad + B_1\, ||u_0||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + B_2\, ||u_0||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} + 2\sqrt{\frac{2}{\alpha_e}}\big(B_1 K_3(u,T) + B_2 K_1(u,T)\big)\Bigg), \\
C_2 &= \sqrt{\frac{2}{\alpha_e}}\, K_2(u,T), \\
C_3 &= \frac{\alpha_c^2}{\alpha_e}\Bigg(A_1 \max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot)||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + A_2 \max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot)||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \\
&\quad + A_1\, ||u_0||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + A_2\, ||u_0||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} + 2\sqrt{\frac{2}{\alpha_e}}\big(A_1 K_3(u,T) + A_2 K_1(u,T)\big)\Bigg),
\end{aligned}$$

where constants $K_1(u,T)$, $K_2(u,T)$, $K_3(u,T)$ are defined by (37)–(39) and constants A_1, A_2, B_1, B_2 are given in Theorem 2.

Proof. This result is obtained by following the proof of Theorem 3 and by using the elliptic error estimate satisfied by the solution of the steady-state model given by Theorem 2. □

Corollary 2. Under the same assumptions as in Theorem 3 and using the same notations, the a priori error estimate as a function of M and N_ξ can be written as

$$\max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot) - u^m_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq C_1\, h + C_2\, \Delta t + C_3\, \left(\frac{1+\log(M)}{\log(N_\xi)}\right)^k. \tag{54}$$

Theorem 5. Under the same assumptions as in Theorem 3 with a quasi-uniform spatial discretization, consider the θ-scheme (29) with $\theta \in [0, 1/2[$. Under the following restrictions on the time-step,

$$\frac{\Delta t}{h^2} \leq \frac{2\alpha_e(C_{\mathcal{D}}^2+1) - 4\epsilon^2 C_{\mathcal{D}}^2}{(C_{\mathcal{D}}^2+1)(1-2\theta)\alpha_c^2 (C_i^*)^2(1+\epsilon)},\ 0 < \epsilon \leq \left(\frac{\alpha_e(1+C_{\mathcal{D}}^2)}{2C_{\mathcal{D}}^2}\right)^{1/2}, \tag{55}$$

where C_i^* is defined by (C.4), the a priori error estimates (36) or (54) hold with

$$\begin{aligned}
C_1 &= \frac{C^*\alpha_c}{\alpha_e}\Bigg(\max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot)||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} + ||u_0||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} + 2\sqrt{c_\epsilon}\, K_1(u,T)\Bigg), \\
C_2 &= \sqrt{2c_\epsilon}\, K_2(u,T), \\
C_3 &= \frac{C(M)\alpha_c}{\alpha_e}\Bigg(\max_{m=1,\dots,N_t} ||u(\cdot,t^m\,;\cdot)||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + ||u_0||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + 2\sqrt{c_\epsilon}\, K_3(u,T)\Bigg).
\end{aligned}$$

The constant c_ϵ is given by $c_\epsilon = (1-2\theta)(1+1/\epsilon)\Delta t + 1/4\epsilon^2$ and $K_1(u,T)$, $K_2(u,T)$, $K_3(u,T)$ are defined by (37)–(39).

Remark 4.1. When considering spatial domains with smooth boundary $\partial\mathcal{D}$, faster convergence rates can be obtained using higher-order FE approximations. Consider $\mathcal{S} = H^k(\Gamma)$, $\mathcal{V} = H_0^1(\mathcal{D}) \cap H^{l+1}(\mathcal{D})$ with $l \geq 1$ and let $\mathcal{V}^h$ be the subspace spanned by piecewise polynomials of degree at most l. Then it holds:

$$||\eta_{h,p_\xi}^m||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c}{\alpha_e}\Big(C(M)\, p_\xi^{-k}\, ||u(\cdot,t^m\,;\cdot)||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} \,+\, \widetilde{C^*}\, h^l\, ||u(\cdot,t^m\,;\cdot)||_{H^{l+1}(\mathcal{D})\otimes L^2(\Gamma)}\Big),$$

where $\widetilde{C^*}$ is a constant independent of h. As a consequence, the error estimate (36) becomes

$$\max_{m=1,\ldots,N_t} ||u(\cdot,t^m\,;\cdot) - u_{h,p_\xi}^m||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq C_1\, h^l + C_2\, \Delta t + C_3\, p_\xi^{-k}, \tag{56}$$

where C_1 is now defined as

$$C_1 = \frac{\widetilde{C^*}\alpha_c}{\alpha_e}\Big(\max_{m=1,\ldots,N_t} ||u(\cdot,t^m\,;\cdot)||_{H^{l+1}(\mathcal{D})\otimes L^2(\Gamma)} + ||u_0||_{H^{l+1}(\mathcal{D})\otimes L^2(\Gamma)} \,+\, \frac{2}{\sqrt{\alpha_e}}\widetilde{K}_1(u,T)\Big),$$

with

$$\widetilde{K}_1(u,T) = \left(\int_0^T \left|\left|\frac{\partial u}{\partial t}(\cdot,s\,;\cdot)\right|\right|^2_{H^{l+1}(\mathcal{D})\otimes L^2(\Gamma)} ds\right)^{1/2}$$

Remark 4.2. The error estimates (36), (54), or (56) hold for second-order parabolic SPDEs defined in Appendix A (see Eqs. (A.1) and (A.2)) since the bilinear form is continuous and elliptic with respect to the norm $||\cdot||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$.

5. APPLICATION TO STOCHASTIC DIFFUSION MODELS

In this section, we shall apply the general error estimates derived earlier to the steady-state and time-dependent stochastic diffusion equations. We shall show how the *a priori* error estimates derived earlier can be sharpened for both cases. Finally, we shall consider a one-dimensional parabolic SPDE model to gain insights into the relationships between the time-step restriction and the input random field parameters.

5.1 Stochastic Diffusion Model

In this section we consider the following stochastic diffusion model [10, 13]

$$\begin{aligned} \frac{\partial u(\mathbf{x},t\,;\boldsymbol{\xi})}{\partial t} - \operatorname{div}\big(\kappa(\mathbf{x}\,;\boldsymbol{\xi})\nabla u(\mathbf{x},t\,;\boldsymbol{\xi})\big) &= f(\mathbf{x},t\,;\boldsymbol{\xi}) && \text{a.s. in } \mathcal{D}\times[0,T]\times\Omega,\\ u(\mathbf{x},t\,;\boldsymbol{\xi}) &= g(\mathbf{x}) && \text{a.s. on } \partial\mathcal{D}\times[0,T]\times\Omega,\\ u(\mathbf{x},0\,;\boldsymbol{\xi}) &= u_0(\mathbf{x}\,;\boldsymbol{\xi}) && \text{a.s. on } \mathcal{D}\times\Omega, \end{aligned} \tag{57}$$

where κ is a random diffusivity field which is strictly positive and bounded, i.e.,

$$0 < k_1 \leq \kappa(\mathbf{x}\,;\boldsymbol{\xi}) \leq k_2 \;\; \text{a.s. in } \mathcal{D}\times\Omega. \tag{58}$$

The source term f is assumed to satisfy

$$\int_\Gamma \int_{\mathcal{D}} \int_0^T |f(\mathbf{x},t\,;\boldsymbol{\xi})|^2 \rho(\boldsymbol{\xi})\, d\boldsymbol{\xi} d\mathbf{x} dt < +\infty, \tag{59}$$

which implies $\int_{\mathcal{D}}\int_0^T |f(\mathbf{x},t\,;\boldsymbol{\xi})|^2\, d\mathbf{x}dt < +\infty$ a.s. in Ω. We also assume that the random field κ is measurable with respect to the σ-algebra $\mathcal{B}(\mathcal{D})\otimes\mathcal{F}$ where $\mathcal{B}(\mathcal{D})$ denotes the σ-algebra associated with $\mathcal{D}$, and that the source term f is measurable with respect to the σ-algebra $\mathcal{B}(\mathcal{D}\times[0,T])\otimes\mathcal{F}$. Models such as (57) are typically used for chemical transport and fluid flow in heterogeneous random media, heat transfer, oil reservoir, and water resources modeling.

The steady-state version of (57) has been extensively studied in the literature; see, for example [7, 11, 14, 15]. The existence and uniqueness of the solution of (57) in $L^2(0,T\,;H_0^1(\mathcal{D}))\otimes L^2(\Gamma)$ is guaranteed by the assumptions (58) and (59); see [13, 34]. The random diffusivity can be discretized using a Karhunen-Loève expansion scheme [28] and written in terms of a finite number of random variables as

$$\kappa(\mathbf{x}\,;\boldsymbol{\xi}) = \kappa^0(\mathbf{x}) + \sum_{i=1}^{M} \xi_i \kappa^i(\mathbf{x}), \tag{60}$$

where ξ_i are uncorrelated uniform random variables (in $[-1,1]$) and $(\kappa^i)_{i\geq 1}$ forms an orthogonal basis of $L^2(\mathcal{D})$. We next consider the following assumption [11] satisfied by the functions κ^0 and κ^i:

$$\sum_{i\geq 1} ||\kappa^i||_{L^\infty(D)} \leq \frac{\lambda}{1+\lambda}\bar{\kappa}^0_{min} \tag{61}$$

with $\bar{\kappa}^0_{min} = \min_{\mathbf{x}\in\bar{D}} \kappa^0(\mathbf{x}) > 0$ and $\lambda > 0$. Note that assumption (61) implies assumption (58) with the lower bound $k_1 = \bar{\kappa}^0_{min}/(1+\lambda)$, and ensures that the series $\sum_{i\geq 1}\xi_i\kappa^i(\mathbf{x})$ is absolutely and uniformly convergent on $D\times\Omega$ (see [11] for more details).

5.2 A Priori Error Estimates: Elliptic Case

In this section we consider the steady-state version of (57). We assume that (58) and (61) hold and that the source term satisfies

$$\int_\Gamma \int_{\mathcal{D}} |f(\mathbf{x}\,;\boldsymbol{\xi})|^2 \rho(\boldsymbol{\xi})\, d\boldsymbol{\xi} d\mathbf{x} < +\infty. \tag{62}$$

Note that the assumption (62) implies $\sup_{\boldsymbol{\xi}\in\Gamma} ||f(\cdot\,;\boldsymbol{\xi})||_{L^2(\mathcal{D})} < +\infty$. The weak formulation is given by

Find $u \in H_0^1(\mathcal{D})\otimes L^2(\Gamma)$ such that:

$$\begin{aligned} A(u,v) &:= \int_\Gamma \int_{\mathcal{D}} \kappa(\mathbf{x}\,;\boldsymbol{\xi})\nabla u(\mathbf{x}\,;\boldsymbol{\xi})\nabla v(\mathbf{x}\,;\boldsymbol{\xi})\rho(\boldsymbol{\xi})\, d\boldsymbol{\xi} d\mathbf{x} \\ &= \int_\Gamma \int_{\mathcal{D}} f(\mathbf{x}\,;\boldsymbol{\xi}) v(\mathbf{x}\,;\boldsymbol{\xi})\rho(\boldsymbol{\xi})\, d\boldsymbol{\xi} d\mathbf{x},\ \forall v \in H_0^1(\mathcal{D})\otimes L^2(\Gamma). \end{aligned} \tag{63}$$

As an illustration of Theorem 1 we shall provide an a priori error estimate for $||u - u_{h,p_{\boldsymbol{\xi}}}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$.

Theorem 6. Consider a random source term $f \in L^2(\mathcal{D})\otimes H^k(\Gamma)$. Let u be the solution of (63) with $u \in (H_0^1(\mathcal{D})\cap H^2(\mathcal{D}))\otimes H^k(\Gamma)$ and $u_{h,p_{\boldsymbol{\xi}}} \in \mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$ denote the FE approximate solution. Then

$$||u - u_{h,p_{\boldsymbol{\xi}}}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c}{\alpha_e}\left(C(M) D_r p_{\boldsymbol{\xi}}^{-k} + C^* C_r h ||f||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \right), \tag{64}$$

where the constant C_r depends on $\mathcal{D}$ and $\left|\left|D_{\mathbf{x}}^{\boldsymbol{\alpha}}\kappa\right|\right|_{L^\infty(\mathcal{D}\times\Gamma)}$, $|\boldsymbol{\alpha}|\leq 1$, while the constant D_r depends on the Poincaré's constant $C_{\mathcal{D}}$, k_1, $(b_j)_{j=1,\ldots M}$ with $b_j = (||\kappa^j||_{L^\infty(\mathcal{D})})/k_1$, and on $\left|\left|D_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} f\right|\right|_{L^2(\mathcal{D})\otimes L^\infty(\Gamma)}$, $|\boldsymbol{\beta}|\leq k$.

Proof. First, we show that $u \in H^1(\mathcal{D})\otimes H^k(\Gamma)$. Following the ideas of Cohen et al. (see [11], Theorem 4.1 therein), it can be shown that

$$\sup_{\boldsymbol{\xi}\in\Gamma} \left|\left|D_{\boldsymbol{\xi}}^{\boldsymbol{\beta}} u(\cdot\,;\boldsymbol{\xi})\right|\right|_{H_0^1(\mathcal{D})} \leq E_{\boldsymbol{\beta}},\ \forall \boldsymbol{\beta}, \tag{65}$$

where the constant $E_{\boldsymbol{\beta}}$ is defined as

$$E_{\boldsymbol{\beta}} = C_{0,\boldsymbol{\beta}} + C_{0,0}|\boldsymbol{\beta}|!\mathbf{b}^{\boldsymbol{\beta}} + \sum_{j,\beta_j\neq 0} \beta_j b_j C_{0,\boldsymbol{\beta}-e_j} + \cdots$$

with $C_{0,\boldsymbol{\beta}} = (C_{\mathcal{D}}/k_1)\sup_{\boldsymbol{\xi}\in\Gamma}\left|\left|D^{\boldsymbol{\beta}}_{\boldsymbol{\xi}} f(\cdot\,;\boldsymbol{\xi})\right|\right|_{L^2(\mathcal{D})}$, $\mathbf{b}^{\boldsymbol{\beta}} = \prod_{j=1}^{M} b_j^{\beta_j}$, and where e_j denotes the Kronecker sequence with the value 1 at the j-th position and 0 elsewhere. Since $f \in L^2(\mathcal{D})\otimes H^k(\Gamma)$, $E_{\boldsymbol{\beta}} < +\infty$ for every $|\boldsymbol{\beta}| \le k$, which means that $u \in H^1_0(\mathcal{D})\otimes H^k(\Gamma)$ or equivalently $u \in H^1(\mathcal{D})\otimes H^k(\Gamma)$ from Poincaré's inequality. Then, from (65) we have

$$||u||^2_{H^1(\mathcal{D})\otimes H^k(\Gamma)} = \sum_{0\le|\boldsymbol{\beta}|\le k}\int_\Gamma ||D^{\boldsymbol{\beta}}_{\boldsymbol{\xi}} u(\cdot\,;\boldsymbol{\xi})||^2_{H^1(\mathcal{D})}\rho(\boldsymbol{\xi})d\boldsymbol{\xi} \le (1+C^2_{\mathcal{D}})\sum_{0\le|\boldsymbol{\beta}|\le k}\int_\Gamma ||D^{\boldsymbol{\beta}}_{\boldsymbol{\xi}} u(\cdot\,;\boldsymbol{\xi})||^2_{H^1_0(\mathcal{D})}\rho(\boldsymbol{\xi})d\boldsymbol{\xi}$$
$$\le (1+C^2_{\mathcal{D}})\sum_{0\le|\boldsymbol{\beta}|\le k} E^2_{\boldsymbol{\beta}} = (D_r)^2. \tag{66}$$

Next, using the following regularity estimate from the analysis of Babuška et al. [7],

$$||u(\cdot\,;\boldsymbol{\xi})||_{H^2(\mathcal{D})} \le C_r||f(\cdot\,;\boldsymbol{\xi})||_{L^2(\mathcal{D})},\ \forall\boldsymbol{\xi}\in\Gamma,$$

we have

$$||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \le C_r||f||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \tag{67}$$

Combining (66), (67), and the estimate (20), we obtain the inequality (64). □

Remark 5.1. Estimating $p_{\boldsymbol{\xi}}^{-k}$ in (64) as in Corollary 1, it can be seen that the stochastic discretization error tends to zero when $M \to +\infty$ and $p_{\boldsymbol{\xi}} \to +\infty$. This trend is what one would expect when using KL expansions for discretization of the SPDE coefficients. However, to proceed further with the analysis derived in the present paper when considering SPDEs with KL expansions, the error arising from random field discretization would need to be taken into account (as done, for example, in [7]).

Remark 5.2. For spatial domains with smooth boundary $\partial\mathcal{D}$, faster convergence rates can be obtained using higher-order FE basis functions. When considering FE piecewise polynomials of degree at most l, the error corresponding to the steady-state stochastic diffusion model scales as $\mathcal{O}(p_{\boldsymbol{\xi}}^{-k} + h^l)$ in Theorem 6.

5.3 Remarks on Time-Step Restriction

We discuss here the time-step restriction (55) when considering the θ-scheme (29) with $\theta \in [0, 1/2[$ for solving the time-dependent diffusion model (57). First, from (58) and using Poincaré's inequality, we have

$$\frac{k_1}{1+C^2_{\mathcal{D}}}||u||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \le A(u,u) \le k_2||u||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)},$$

meaning that $\alpha_e = k_1/(1+C^2_{\mathcal{D}})$ and $\alpha_c = k_2$. To proceed further, we shall express k_1 and k_2 in terms of the random field parameters. To illustrate, we consider $\mathcal{D} = [0,1] \subset \mathbb{R}$ with the covariance function defined as $C(x,y) = \sigma^2\exp(-|x-y|/L_c)$, where σ and L_c denote the standard deviation and the correlation length, respectively. The functions $(\kappa^i)_{i\ge1}$ in (60) are given by $\kappa^i(x) = \sqrt{\lambda_i}c^i(x)$, where $\{\lambda_i, c^i\}$ are the eigenvalues and eigenfunctions of the Fredholm equation

$$\int_0^1 C(x,y)c^i(y)dy = \lambda_i c^i(x). \tag{68}$$

From the definition of C, the analytical solution of (68) is explicitly given by [1, 2]

$$c^i(x) = \begin{cases} \dfrac{\cos(\omega_i(x-0.5))}{\sqrt{0.5+\sin(\omega_i)/2\omega_i}} & \text{i even,} \\[2ex] \dfrac{\sin(\omega_i(x-0.5))}{\sqrt{0.5-\sin(\omega_i)/2\omega_i}} & \text{i odd,} \end{cases}$$

$$\lambda_i = \sigma^2 \frac{2L_c}{1+(\omega_i L_c)^2}, \tag{69}$$

where ω_i are positive roots of $(1 - L_c\omega\tan(\omega/2))(L_c\omega + \tan(\omega/2))$. It is to be noted that the decay rate of λ_i increases as L_c becomes smaller (see [1]). Since c^i are bounded functions, $\xi_i \in [-1, 1]$ and from (60), (69), it follows that

$$\alpha_c = k_2 = \sup_{(x,\xi)\in\mathcal{D}\times\Gamma} \kappa(x\,;\xi) = ||\kappa^0||_{L^\infty(\mathcal{D})} + \sigma\sqrt{2L_c}\sum_{i=1}^{M} \frac{||c^i||_{L^\infty(\mathcal{D})}}{\sqrt{1+(\omega_i L_c)^2}}. \tag{70}$$

We are now in a position to discuss the influence of the random field parameters (L_c and σ) on the allowable time-step Δt for the case when $\theta \in [0, 1/2[$. From (70), it can be seen that α_c increases when σ becomes larger. As a result of this, the allowable interval for the time-step Δt shrinks [see Eq. (55)]. Conversely, α_c decreases when L_c takes larger values [see Eq. (70)]. Therefore the upper bound (55) on the time-step is less restrictive, meaning that larger values of Δt can be considered. The latter statement is coherent with the fact that when L_c goes to $+\infty$, the random field κ tends to the averaged deterministic field κ^0, meaning that Δt is not restricted by the random field parameters anymore.

6. A NOTE ON A PRIORI ERROR ESTIMATES FOR FUNCTIONAL APPROXIMATIONS

6.1 Preliminaries

In many problems of practical interest, one is concerned by evaluating a quantity of interest given by a functional $J(u)$, where u is the solution of a SPDE model. A question which arises naturally concerns the estimation of the error $|J(u)-J(u_{h,p_\xi})|$, where u_{h,p_ξ} is a stochastic PC Galerkin approximation of u. In this context, there exists a wide body of literature on a posteriori error estimation for deterministic PDE models. For example, computable a posteriori error estimates for FE approximations of deterministic elliptic PDEs are derived in [35, 36]. Pierce and Giles [29] presented an approach for improved approximation of functionals depending on linear or nonlinear deterministic PDE solutions using adjoint methods. A detailed overview of adjoint methods for a posteriori error analysis of FE approximations of deterministic functionals is provided in [37]. A posteriori error estimates for stochastic FE methods based on an adjoint formulation and mesh refinement procedures are studied in [38]. Computable a posteriori error estimates based on a stochastic Galerkin projection scheme for the adjoint problem are provided in [30].

In this section, we shall consider adjoint-based corrections to functionals computed using the approach originally developed by Pierce and Giles [29] and extended to SPDEs by Butler et al. [30]. Our main objective is to demonstrate that this recovery scheme is *superconvergent* based on the a priori error estimates derived in the earlier sections for elliptic SPDEs. Our analysis follows the work of Giles and Süli [37] dealing with deterministic PDEs (see Theorem 7.1 therein). For clarity of exposition, let us first recall the *primal* and *dual* SPDE (strong) formulations which are, respectively, given by

$$\mathcal{L}_\xi u(\mathbf{x}\,;\boldsymbol{\xi}) = f(\mathbf{x}\,;\boldsymbol{\xi}) \text{ a.s. in } \mathcal{D}\times\Omega, \tag{71}$$

$$\mathcal{L}_\xi w(\mathbf{x}\,;\boldsymbol{\xi}) = g(\mathbf{x}\,;\boldsymbol{\xi}) \text{ a.s. in } \mathcal{D}\times\Omega, \tag{72}$$

where $\mathcal{L}_\xi$ will be assumed to be a randomly parametrized second-order self-adjoint differential operator and $f, g \in L^2(\mathcal{D}) \otimes L^2(\Gamma)$ are given random source terms. For the simplicity of presentation, (71) and (72) are supplemented with homogeneous Dirichlet boundary conditions. The weak form corresponding to the primal problem is written as
Find $u \in \mathcal{V}\otimes\mathcal{S}$ such that:

$$A(u,v) = (f,v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall v \in \mathcal{V}\otimes\mathcal{S}, \tag{73}$$

with $\mathcal{V} = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$ and $\mathcal{S} = H^k(\Gamma)$. As assumed earlier, the bilinear form A is α_c-continuous and α_e-elliptic with respect to the norm $||\cdot||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$. The weak form associated with the dual problem is given by
Find $w \in \mathcal{V}\otimes\widetilde{\mathcal{S}}$ such that:

$$A(v,w) = (g,v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall v \in \mathcal{V}\otimes\widetilde{\mathcal{S}}, \tag{74}$$

where $\widetilde{\mathcal{S}} = H^l(\Gamma)$. Strictly speaking, the Sobolev index l can be different from k, depending on the stochastic regularity of the source term g.

We now focus on the definition of the functional $J(u)$ which clearly depends on the application under consideration. In the present analysis, we shall consider stochastic functionals of the form

$$J(u) = (g(\cdot\,;\boldsymbol{\xi}), u(\cdot\,;\boldsymbol{\xi}))_{L^2(\mathcal{D})}, \tag{75}$$

where u is the solution of the SPDE model (71) and g is the source term of (72). Stochastic pointwise functionals have been studied in [39] when considering linear random algebraic equations, for which an error analysis with respect to the norm $||\cdot||_{L^\infty(\Gamma)}$ is provided. However, the convergence result derived therein is obtained by *assuming* a particular form of primal/dual approximation errors in the L^∞ norm.

In the present work, our aim is to leverage the a priori error estimates derived earlier for elliptic SPDEs. To begin with, consider error analysis for the approximation of (75) with respect to the norm $||\cdot||_{L^2(\Gamma)}$. Using the Cauchy-Schwarz inequality, it can be shown that

$$||J(u) - J(u_{h,p_{\boldsymbol{\xi}}})||_{L^2(\Gamma)} \leq \sup_{\boldsymbol{\xi}\in\Gamma} ||g(\cdot\,;\boldsymbol{\xi})||_{L^2(\mathcal{D})} ||u - u_{h,p_{\boldsymbol{\xi}}}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}.$$

The a priori error estimates (given by Theorem 1 or Theorem 2) can be used in the preceding inequality to obtain convergence rates for functionals approximated using the solution of the primal SPDE. However, the a priori error estimates derived earlier for elliptic SPDEs cannot be used further in this framework when considering improved functionals based on the solution of an adjoint/dual problem. In this approach, instead of considering $J(u_{h,p_{\boldsymbol{\xi}}})$ as an approximation of the exact functional $J(u)$, improved functionals are defined by means of an adjoint correction term; see Section 6.3. It is to be noted that similar issues also occur for pointwise functionals depending on a deterministic PDE solution. As discussed in Giles and Süli [37], pointwise functionals cannot be accommodated within the framework of Hilbertian error analysis. To circumvent this technical problem, one might carry out the error analysis in a reflexive Banach setting, or, alternatively, define another type of functional based on a local average over Γ, following the ideas in [37]. Since our a priori error estimates for elliptic SPDEs are derived in an Hilbertian framework, we choose the second option to proceed further. In other words, instead of directly working with the stochastic functional (75), we shall consider a linear output functional given by a local average of (75) in a ball defined over Γ, i.e.,

$$J(u) = \frac{1}{|\mathcal{B}(\hat{\boldsymbol{\xi}}, r)|}(g, u)_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))}, \tag{76}$$

where $\mathcal{B}(\hat{\boldsymbol{\xi}}, r) \subset \Gamma$ denotes a ball of radius r centered at a given random vector $\hat{\boldsymbol{\xi}}$, with $|\mathcal{B}(\hat{\boldsymbol{\xi}}, r)| = \int_{\mathcal{B}(\hat{\boldsymbol{\xi}},r)} \rho(\boldsymbol{\xi}) d\boldsymbol{\xi}$. In the following sections, we shall provide error estimates for primal and adjoint-based corrected approximations of the preceding functional.

6.2 A Priori Error Estimates for Primal Solution Based Functional Approximations

We shall first provide a priori error estimates for the approximation $J(u_{h,p_{\boldsymbol{\xi}}})$ based on the solution of the primal problem before examining the error associated with the adjoint-based corrected approximation. In what follows, the approximation space for the primal problem is given by $\mathcal{V}^h \otimes \mathcal{S}^{p_{\boldsymbol{\xi}}}$, where h is the spatial discretization mesh spacing parameter and $p_{\boldsymbol{\xi}}$ is the PC approximation order. By definition of J and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |J(u) - J(u_{h,p_{\boldsymbol{\xi}}})| &\leq \frac{1}{|\mathcal{B}(\hat{\boldsymbol{\xi}}, r)|} ||g||_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} ||u - u_{h,p_{\boldsymbol{\xi}}}||_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} \\ &\leq \frac{1}{|\mathcal{B}(\hat{\boldsymbol{\xi}}, r)|} ||g||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ||u - u_{h,p_{\boldsymbol{\xi}}}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \end{aligned} \tag{77}$$

Applying Theorem 1 for estimating the primal error approximation, we get the following error estimate:

$$|J(u) - J(u_{h,p_\xi})| \leq \frac{\alpha_c}{\alpha_e} \frac{1}{|\mathcal{B}(\hat{\xi}, r)|} ||g||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$$
$$\times \left(C(M)\, p_\xi^{-k}\, ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \right). \tag{78}$$

Under additional regularity assumptions for the dual problem (72), i.e.,

$$||w||_{H^2(\mathcal{D})\otimes L^2(\Gamma)} \leq C_r ||g||_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ ||w||_{H^1(\mathcal{D})\otimes H^l(\Gamma)} \leq \widetilde{C}_r ||g||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}, \tag{79}$$

a sharper functional error estimate can be obtained using a a slightly different version of Theorem 2 for estimating $||u - u_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ given below:

$$||u - u_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{\alpha_c^2}{\alpha_e} \left(C_1(M)\, p_\xi^{-s} + C_2(M)\, h^2 \right),$$

where $s = \min\{k + l, 2k, 2l\}$ and C_1, C_2 are constants independent of h and p_ξ given by

$$C_1(M) = \left(C(M)^2 \widetilde{C}_r + \frac{C(M) C^* C_r}{2} \right) ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + \frac{C(M) C^* \widetilde{C}_r}{2} ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)},$$

$$C_2(M) = \frac{C(M) C^* C_r}{2} ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + \left((C^*)^2 C_r + \frac{C(M) C^* \widetilde{C}_r}{2} \right) ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}.$$

Hence we get the following a priori error estimate:

$$|J(u) - J(u_{h,p_\xi})| \quad \leq \quad \frac{\alpha_c^2}{\alpha_e} \frac{1}{|\mathcal{B}(\hat{\xi}, r)|} ||g||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \left(C_1(M)\, p_\xi^{-s} + C_2(M)\, h^2 \right). \tag{80}$$

It is worth noting that superconvergence of functional approximations automatically holds when considering deterministic Galerkin FE methods for computing the primal and dual solutions (see [29] where deterministic linear functionals depending on the solution of deterministic PDE models are studied). In the present analysis, where linear output functionals that depend on the solution of SPDE models are considered, we also obtain (under regularity assumptions for the adjoint problem) a superconvergence result [see Eq. (80)], as expected when using Galerkin projection schemes.

6.3 A Priori Error Estimates for Adjoint-Based Corrected Functional Approximations

We first introduce the finite-dimensional approximation subspaces that are needed in our derivation, i.e., $\mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$ for the primal problem (71) and $\mathcal{V}^H \otimes \mathcal{S}^{q_\xi}$ for the dual problem (72). At this stage, there is no restriction on the definition of the mesh-sizes (h, H) and the PC orders (p_ξ, q_ξ) used for solving the primal and dual problems. The Galerkin approximation of the dual formulation,

$$A(v_{H,q_\xi}, w_{H,q_\xi}) = (g, v_{H,q_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall v_{H,q_\xi} \in \mathcal{V}^H \otimes \mathcal{S}^{q_\xi}, \tag{81}$$

where $\mathcal{V}^H$ and $\mathcal{S}^{q_\xi}$ are given by (11) and (12), respectively, can be formally written in an operator form as

$$\mathcal{L}_\xi w_{H,q_\xi} = g_{H,q_\xi} = g + z_{H,q_\xi}, \tag{82}$$

where z_{H,q_ξ} belongs to the orthogonal complement of $\mathcal{V}^H \otimes \mathcal{S}^{q_\xi}$. We are now in a position to write the following splitting:

$$J(u) = \frac{1}{|\mathcal{B}(\hat{\xi}, r)|} \Big((g, u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\xi},r))} - (g_{H,q_\xi}, u_{h,p_\xi} - u)_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\xi},r))}$$
$$+ (g_{H,q_\xi} - g, u_{h,p_\xi} - u)_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\xi},r))} \Big). \tag{83}$$

Using the approximate dual problem (82), the second term in the previous splitting is written as

$$\begin{aligned}(g_{H,q_\xi}, u - u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} &= \int_{\mathcal{B}(\hat{\boldsymbol{\xi}},r)} \left(\mathcal{L}_{\boldsymbol{\xi}} w_{H,q_\xi}, u - u_{h,p_\xi}\right)_{L^2(\mathcal{D})} \rho(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \int_{\mathcal{B}(\hat{\boldsymbol{\xi}},r)} \left(w_{H,q_\xi}, \mathcal{L}_{\boldsymbol{\xi}}(u - u_{h,p_\xi})\right)_{L^2(\mathcal{D})} \rho(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= (w_{H,q_\xi}, f - \mathcal{L}_{\boldsymbol{\xi}} u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))}.\end{aligned}$$

This suggests the definition of the improved functional,

$$J_{imp}(u_{h,p_\xi}, w_{H,q_\xi}) = J(u_{p_\xi}) + \frac{1}{|\mathcal{B}(\hat{\boldsymbol{\xi}},r)|}(w_{H,q_\xi}, f - \mathcal{L}_{\boldsymbol{\xi}} u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))}, \tag{84}$$

where the adjoint correction term $1/|\mathcal{B}(\hat{\boldsymbol{\xi}},r)|(w_{H,q_\xi}, f - \mathcal{L}_{\boldsymbol{\xi}} u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))}$ is a computable quantity depending on the dual solution and the primal residual. The adjoint correction can be interpreted as a computable a posteriori error estimate [30] or as a correction term for improving the primal solution-based approximation [39].

We shall now provide an a priori error estimate for the improved adjoint corrected functional given by (84).

Theorem 7. Let J and J_{imp} denote the exact and improved functionals given by (76) and (84), respectively. Then it holds

$$\begin{aligned}|J(u) - J_{imp}(u_{h,p_\xi}, w_{H,q_\xi})| \leq \frac{1}{|\mathcal{B}(\hat{\boldsymbol{\xi}},r)|}\frac{\alpha_c^3}{\alpha_e^2}&\left(C(M)\, p_{\boldsymbol{\xi}}^{-k}\, ||u||_{H^1(\mathcal{D})\otimes H^k(\Gamma)} + C^* h\, ||u||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\right) \\ &\times \left(C(M)\, q_{\boldsymbol{\xi}}^{-l}\, ||w||_{H^1(\mathcal{D})\otimes H^l(\Gamma)} + C^* H\, ||w||_{H^2(\mathcal{D})\otimes L^2(\Gamma)}\right),\end{aligned} \tag{85}$$

where $u_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$ and $w_{H,q_\xi} \in \mathcal{V}^H \otimes \mathcal{S}^{q_\xi}$. The constants C and C^* are independent of $p_{\boldsymbol{\xi}}$, $q_{\boldsymbol{\xi}}$, h, and H.

Proof. By construction, we have

$$\begin{aligned}J(u) &= J_{imp}(u_{h,p_\xi}, w_{H,q_\xi}) + \frac{1}{|\mathcal{B}(\hat{\boldsymbol{\xi}},r)|}(g_{H,q_\xi} - g, u_{h,p_\xi} - u)_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} \\ &= J_{imp}(u_{h,p_\xi}, w_{H,q_\xi}) + \frac{1}{|\mathcal{B}(\hat{\boldsymbol{\xi}},r)|}(w_{H,q_\xi} - w, \mathcal{L}_{\boldsymbol{\xi}}(u_{h,p_\xi} - u))_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))},\end{aligned} \tag{86}$$

using the self-adjointness of $\mathcal{L}_{\boldsymbol{\xi}}$. For the sake of illustration, consider the case when $\mathcal{L}_{\boldsymbol{\xi}}$ is given by a randomly parametrized second-order differential operator defined by (A.1). Applying integration by parts over $\mathcal{D}$, we get

$$\begin{aligned}(w_{H,q_\xi} - w, \mathcal{L}_{\boldsymbol{\xi}}(u_{h,p_\xi} - u))_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} &= \sum_{i,j=1}^{d}\left(a_{ij}\frac{\partial}{\partial x_j}(u_{h,p_\xi} - u), \frac{\partial}{\partial x_i}(w_{H,q_\xi} - w)\right)_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} \\ &+ \sum_{i=1}^{d}\left(b_i\frac{\partial}{\partial x_i}(u_{h,p_\xi} - u), w_{H,q_\xi} - w\right)_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} \\ &+ (c(u_{h,p_\xi} - u), w_{H,q_\xi} - w)_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))}.\end{aligned}$$

Using the same arguments as in Appendix A, we obtain

$$\begin{aligned}&\left|(w_{H,q_\xi} - w, \mathcal{L}_{\boldsymbol{\xi}}(u_{h,p_\xi} - u))_{L^2(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))}\right| \\ &\leq \alpha_c\, ||u - u_{h,p_\xi}||_{H_0^1(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))}\, ||w - w_{H,q_\xi}||_{H_0^1(\mathcal{D})\otimes L^2(\mathcal{B}(\hat{\boldsymbol{\xi}},r))} \\ &\leq \alpha_c\, ||u - u_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}\, ||w - w_{H,q_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}\end{aligned} \tag{87}$$

where α_c is the continuity constant of the bilinear form A,given here by $\alpha_c = \lambda_a^{\max} + \max_{i=1,\dots d} |\bar{b}_i|\sqrt{d}\, C_{\mathcal{D}} + \bar{c}\, C_{\mathcal{D}}^2$. Using (86) and applying Theorem 1 to estimate $||u - u_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$ and $||w - w_{H,q_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$ in the right-hand side of (87), we get the final result. □

Remark 6.1. In the limiting case when $\mathcal{B}(\hat{\xi}, r) = \Gamma$, we have $|\mathcal{B}(\hat{\xi}, r)| = 1$ and J is given by $J(u) = (g, u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$. From the splitting

$$J(u) = (g, u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)} - (g_{H,q_\xi}, u_{h,p_\xi} - u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + (g_{H,q_\xi} - g, u_{h,p_\xi} - u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)},$$

it can be seen that the adjoint correction $(g_{H,q_\xi}, u - u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ will be nonzero only if we consider refined approximate subspaces for solving the dual problem, i.e., $H < h$ and/or $q_\xi > p_\xi$. Indeed, if we consider $H \geq h$ and $q_\xi \leq p_\xi$ so that $\mathcal{V}^H \otimes \mathcal{S}^{q_\xi} \subset \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$, the adjoint correction vanishes due to Galerkin orthogonality—this can be shown as follows:

$$(g_{h,p_\xi}, u - u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = (\mathcal{L}_\xi w_{h,p_\xi}, u - u_{h,p_\xi})_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = A(u - u_{h,p_\xi}, w_{h,p_\xi}) = 0.$$

In summary, an adjoint-based correction is possible in the limiting case only if refined approximation subspaces are used for solving the dual problem. From a computational point of view, this strategy will be inefficient since the algorithmic complexity of the dual and the primal problems tend to be equivalent in practical implementations. For the case when the output functional is given by (76), the Galerkin orthogonality condition does not apply since a local integral over a ball in the random space is used for the adjoint correction term. Hence, the approach based on the improved functional (84) is computationally efficient for computing functionals of the form (76) since a coarse mesh size ($H > h$) and a low PC order ($q_\xi < p_\xi$) can be used for solving the dual problem.

7. CONCLUDING REMARKS

In this paper, we present some a priori error estimates for FE approximation of a class of elliptic and parabolic linear SPDEs in the setting of finite-dimensional noise. In the elliptic case, we derive a priori error estimates that hold under some spatial and stochastic regularity assumptions for the analytical SPDE solution. We also derive a sharper estimate for the convergence rate under additional elliptic regularity assumptions. For the case of parabolic SPDE models, we present a detailed stability analysis of a class of weighted time-stepping schemes. This stability analysis is subsequently used to account for the effect of the temporal discretization error on the convergence rate of stochastic finite element approximations. The results obtained are applied to the steady-state and time-dependent stochastic diffusion equations.

Finally, we consider primal and adjoint-based corrected approximations of linear stochastic functionals that depend on the solution of elliptic SPDEs. We focus on a special case involving local averages of stochastic functionals to gain insights into the convergence rate. The present analysis shows that for stochastic finite element methods based on Galerkin projection, the primal and adjoint-based correction procedures provide superconvergent estimates of a class of linear functionals that depend on the solution of elliptic SPDE models, provided appropriate regularity conditions are satisfied.

The present analysis was limited to Legendre PC expansions and further work is required to extend the analysis to generalized PC expansions (e.g., Hermite, Laguerre, and Jacobi polynomials). Recently, Ernst et al. [40] provided a theoretical analysis of conditions under which generalized PC expansions will converge to the correct limit. It is necessary to extend this work further in order to derive rates of convergence for different types of PC expansions. Numerical studies are also required to compare the theoretical error estimates obtained in the present analysis with empirically obtained convergence rates. Another related topic which remains to be investigated further involves stability analysis of weighted temporal discretization schemes when the SPDE solution is assumed to satisfy polydisc analyticity assumptions in the complex plane.

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APPENDIX A. CONTINUITY AND COERCIVITY CONDITIONS FOR A CLASS OF SECOND-ORDER SPDES

We consider a class of SPDE models (3) when $\mathcal{L}_{\boldsymbol{\xi}}$ is a second-order differential operator of the form

$$\mathcal{L}_{\boldsymbol{\xi}} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x}\,;\boldsymbol{\xi})\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(\mathbf{x}\,;\boldsymbol{\xi})\frac{\partial}{\partial x_i} + c(\mathbf{x}\,;\boldsymbol{\xi}), \tag{A.1}$$

with homogeneous boundary conditions

$$u(\mathbf{x}\,;\boldsymbol{\xi}) = 0 \text{ a.s. on } \partial\mathcal{D}\times\Omega, \tag{A.2}$$

and $f \in L^2(\mathcal{D}\times\Omega)$. We assume that the matrix $\mathbf{a}(\mathbf{x}\,;\boldsymbol{\xi}) : \mathcal{D}\times\Omega \to \mathbb{R}^{d\times d}$ with entries $a_{ij}(\mathbf{x}\,;\boldsymbol{\xi})$ is symmetric positive definite a.s. in $\mathcal{D}\times\Omega$ with its eigenvalues $\lambda_i(\mathbf{x}\,;\boldsymbol{\xi})$ such that

$$0 < \lambda_a^{\min} \le \lambda_i(\mathbf{x}\,;\boldsymbol{\xi}) \le \lambda_a^{\max} < +\infty \text{ a.s. in } \mathcal{D}\times\Omega, i = 1,\dots,d.$$

In addition, let the functions $(b_i)_{i=1,\dots d}$ and c in (A.1) satisfy

$$\begin{aligned} c(\mathbf{x}\,;\boldsymbol{\xi}) - \frac{1}{2}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}b_i(\mathbf{x}\,;\boldsymbol{\xi}) \ge 0 \quad & \text{a.s. in } \mathcal{D}\times\Omega,\\ b_i(\mathbf{x}\,;\boldsymbol{\xi}) \le \bar{b}_i < +\infty \quad & \text{a.s. in } \mathcal{D}\times\Omega,\ i = 1,\dots,d,\\ c(\mathbf{x}\,;\boldsymbol{\xi}) \le \bar{c} < +\infty \quad & \text{a.s. in } \mathcal{D}\times\Omega. \end{aligned}$$

Then the stochastic weak formulation can be written as follows.
Find $u \in H_0^1(\mathcal{D})\otimes L^2(\Gamma)$ such that:

$$\begin{aligned} A(u,v) &= \sum_{i,j=1}^{d}\left(a_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \sum_{i=1}^{d}\left(b_i\frac{\partial u}{\partial x_i}, v\right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + (cu,v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\\ &= (f,v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall v \in H_0^1(\mathcal{D})\otimes L^2(\Gamma). \end{aligned}$$

Since we consider bounded spatial domains, the norms $||\cdot||_{H^1(\mathcal{D})}$ and $||\cdot||_{H_0^1(\mathcal{D})}$ are equivalent from Poincaré's inequality. Hence the continuity and coercivity of A can be proved with respect to the norm $||\cdot||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}$. Since the mapping $\mathbf{u}_1, \mathbf{u}_2 \in (L^2(\mathcal{D}))^d \otimes L^2(\Gamma) \mapsto (\mathbf{a}\mathbf{u}_1, \mathbf{u}_2)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ defines an inner product,[1] we have

$$|(\mathbf{a}\nabla u, \nabla v)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}| \le \lambda_a^{\max}||\nabla u||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}||\nabla v||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = \lambda_a^{\max}||u||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}||v||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}.$$

For the second term, using the Cauchy-Schwarz inequality and noting that $\mathcal{D}$ is bounded, we get

$$\begin{aligned} \left|\sum_{i=1}^{d}\left(b_i\frac{\partial u}{\partial x_i}, v\right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\right| &\le \max_{i=1,\dots d}|\bar{b}_i|\sum_{i=1}^{d}\left|\left|\frac{\partial u}{\partial x_i}\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)}||v||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\\ &\le \max_{i=1,\dots d}|\bar{b}_i|\sqrt{d}\left(\sum_{i=1}^{d}\left|\left|\frac{\partial u}{\partial x_i}\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\right)^{1/2}||v||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\\ &\le \max_{i=1,\dots d}|\bar{b}_i|\sqrt{d}\,C_{\mathcal{D}}\,||u||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}||v||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}, \end{aligned}$$

where $C_{\mathcal{D}}$ denotes Poincaré's constant which only depends on the spatial dimension d and the diameter of $\mathcal{D}$. Applying Poincaré's inequality again for the third term in A, we obtain

$$|A(u,v)| \le \left(\lambda_a^{\max} + \max_{i=1,\dots d}|\bar{b}_i|\sqrt{d}\,C_{\mathcal{D}} + \bar{c}\,C_{\mathcal{D}}^2\right)||u||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}||v||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}.$$

[1]The vectorial inner product defined on $(L^2(\mathcal{D}))^d\otimes L^2(\Gamma)$ is given by $(\mathbf{u}_1,\mathbf{u}_2)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = \sum_{i=1}^{d}\big((\mathbf{u}_1)_i, (\mathbf{u}_2)_i\big)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$. For compactness, we use the notation $(\cdot,\cdot)_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ instead of $(\cdot,\cdot)_{(L^2(\mathcal{D}))^d\otimes L^2(\Gamma)}$ or similarly $||\nabla u||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ instead of $||\nabla u||_{(L^2(\mathcal{D}))^d\otimes L^2(\Gamma)}$ in the paper.

We focus now on proving the coercivity of A. We write

$$\begin{aligned} A(u,u) &= (\mathbf{a}\nabla u, \nabla u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \sum_{i=1}^{d}\left(b_i\frac{\partial u}{\partial x_i}, u\right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + (cu,u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \\ &= (\mathbf{a}\nabla u, \nabla u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \int_\Gamma\int_{\mathcal{D}}\left(cu^2 + \sum_{i=1}^{d} b_i\frac{1}{2}\frac{\partial u^2}{\partial x_i}\right)d\mathbf{x}\,\rho(\boldsymbol{\xi})d\boldsymbol{\xi} \\ &= (\mathbf{a}\nabla u, \nabla u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \int_\Gamma\int_{\mathcal{D}}\left(c - \frac{1}{2}\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i}\right)u^2 d\mathbf{x}\,\rho(\boldsymbol{\xi})d\boldsymbol{\xi} \end{aligned}$$

applying integration by parts to the second term within brackets. The coercivity condition (6) then holds since we have

$$(\mathbf{a}\nabla u, \nabla u)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \geq \lambda_a^{\min}\,||u||^2_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}$$

and $c - (1/2)\sum_{i=1}^{d}(\partial b_i/\partial x_i) \geq 0$ a.s. in $\mathcal{D}\times\Omega$. The boundedness condition (7) directly follows from

$$|l(v)| \leq ||f||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}||v||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq C_{\mathcal{D}}||f||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}||v||_{H_0^1(\mathcal{D})\otimes L^2(\Gamma)}.$$

The bilinear form A then satisfies the assumptions (5), (6), and (7) with respect to the norm $||\cdot||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$.

APPENDIX B. PROOF OF LEMMA 1

In this section we prove Lemma 1 by induction on the number of random variables M. The proof follows [41] where error estimates are given for Legendre polynomial approximations (such error estimates are stated in [42] without a proof; see Theorem 3.1 therein). For the sake of clarity we consider $\mathcal{V} = L^2(\mathcal{D})$, however, the proof can be easily extended to general Sobolev spaces.

Let us start by considering the case when $M = 1$. We assume that $u(\mathbf{x};\cdot) \in H^k(\Gamma)$ with $k = 2m$ (for brevity the case $k = 2m+1$ is not presented here as it is based on the same arguments). From

$$\pi_{p_\xi}(u) := v_{p_\xi}(\mathbf{x};\xi) = \sum_{\alpha=0}^{p_\xi} v_\alpha(\mathbf{x})L_\alpha(\xi),$$

we have

$$||u - v_{p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = \sum_{\alpha\geq p_\xi+1}||v_\alpha||^2_{L^2(\mathcal{D})}||L_\alpha||^2_{L^2(\Gamma)},$$

where v_α is given by (17), i.e.,

$$v_\alpha(\mathbf{x}) = \frac{\int_\Gamma u(\mathbf{x};\xi)L_\alpha(\xi)\rho(\xi)d\xi}{||L_\alpha||^2_{L^2(\Gamma)}}.$$

Using the Legendre operator $\mathcal{L} = -(d/d\xi)[(1-\xi^2)(d/d\xi)]$ such that $\mathcal{L}L_\alpha = \alpha(\alpha+1)L_\alpha$, we get

$$v_\alpha(\mathbf{x}) = \frac{1}{||L_\alpha||^2_{L^2(\Gamma)}}\frac{1}{\alpha(\alpha+1)}\int_\Gamma u(\mathbf{x};\xi)\,\mathcal{L}L_\alpha(\xi)\rho(\xi)d\xi = \frac{1}{||L_\alpha||^2_{L^2(\Gamma)}}\frac{1}{\alpha(\alpha+1)}\int_\Gamma \mathcal{L}u(\mathbf{x};\xi)\,L_\alpha(\xi)\rho(\xi)d\xi$$

since $\mathcal{L}$ is self-adjoint. Iterating m times this result, we obtain

$$v_\alpha(\mathbf{x}) = \frac{1}{||L_\alpha||^2_{L^2(\Gamma)}}\frac{1}{(\alpha(\alpha+1))^m}\int_\Gamma \mathcal{L}^m u(\mathbf{x};\xi)L_\alpha(\xi)\rho(\xi)d\xi$$

and then

$$||v_\alpha||^2_{L^2(\mathcal{D})} = \frac{1}{||L_\alpha||^4_{L^2(\Gamma)}}\frac{1}{(\alpha(\alpha+1))^{2m}}\int_{\mathcal{D}}\left(\int_\Gamma \mathcal{L}^m u(\mathbf{x};\xi)L_\alpha(\xi)\rho(\xi)d\xi\right)^2 d\mathbf{x},$$

leading to

$$
\begin{aligned}
||u - v_{p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} &= \sum_{\alpha \geq p_\xi+1} \frac{1}{||L_\alpha||^4_{L^2(\Gamma)}} \frac{1}{(\alpha(\alpha+1))^{2m}} \int_{\mathcal{D}} \left(\int_\Gamma \mathcal{L}^m u(\mathbf{x}\,;\xi) L_\alpha(\xi) d\xi \right)^2 d\mathbf{x}\ ||L_\alpha||^2_{L^2(\Gamma)} \\
&\leq p_\xi^{-4m} \sum_{\alpha \geq p_\xi+1} \int_{\mathcal{D}} \left(\frac{\int_\Gamma \mathcal{L}^m u(\mathbf{x}\,;\xi) L_\alpha(\xi) \rho(\xi) d\xi}{||L_\alpha||^2_{L^2(\Gamma)}} \right)^2 d\mathbf{x}\ ||L_\alpha||^2_{L^2(\Gamma)} \\
&\leq p_\xi^{-4m} \int_D \sum_{\alpha \geq 0} \left(\frac{\int_\Gamma \mathcal{L}^m u(\mathbf{x}\,;\xi) L_\alpha(\xi) \rho(\xi) d\xi}{||L_\alpha||^2_{L^2(\Gamma)}} \right)^2 ||L_\alpha||^2_{L^2(\Gamma)}\ d\mathbf{x} \\
&= p_\xi^{-4m} \int_{\mathcal{D}} ||\mathcal{L}^m u(\mathbf{x}\,;\cdot)||^2_{L^2(\Gamma)}\ d\mathbf{x}.
\end{aligned}
$$

We then use the property (see Lemma 1.3 in [41]) that the operator $\mathcal{L}^m$ is continuous from $H^{l+2m}(\Gamma)$ into $H^l(\Gamma)$. Taking $l = 0$, there exists a constant $\Lambda > 0$ such that

$$
||\mathcal{L}^m \varphi||_{L^2(\Gamma)} \leq \Lambda ||\varphi||_{H^{2m}(\Gamma)},\ \forall \varphi \in H^{2m}(\Gamma).
$$

Therefore, we have

$$
||u - v_{p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \Lambda^2 p_\xi^{-4m} \int_{\mathcal{D}} ||u(\mathbf{x}\,;\cdot)||^2_{H^{2m}(\Gamma)}\ d\mathbf{x} = \Lambda^2 p_\xi^{-2k} ||u||^2_{L^2(\mathcal{D})\otimes H^k(\Gamma)},
$$

which coincides with the error estimate (19), with $C = \Lambda$.

The next step involves proving the result for $M \geq 2$ random variables. Let us first define the projector (acting on the jth random variable)

$$
\pi^{\xi_j}_{p_\xi}(u) = \sum_{\alpha=0}^{p_\xi} v_\alpha(\mathbf{x}\,;\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_M) L_\alpha(\xi_j),
$$

with

$$
v_\alpha(\mathbf{x}\,;\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_M) = \frac{(u(\mathbf{x}\,;\xi_1, \ldots, \xi_{j-1}, \cdot, \xi_{j+1}, \ldots, \xi_M), L_\alpha)_{L^2(\Gamma_j)}}{||L_\alpha||^2_{L^2(\Gamma_j)}}.
$$

It can be seen that the projector $\pi^{\xi_1}_{p_\xi} \circ \cdots \circ \pi^{\xi_M}_{p_\xi}$ corresponds to the projection onto the M-dimensional chaos subspace with order p_ξ. As an example, consider the case when $M = 2$, $p_\xi = 2$. We have

$$
\begin{aligned}
\pi^{\xi_1}_{p_\xi} \circ \pi^{\xi_2}_{p_\xi}(u) &= \pi^{\xi_1}_{p_\xi} \left(\sum_{\alpha=0}^{p_\xi} v_\alpha(\mathbf{x}\,;\xi_1) L_\alpha(\xi_2) \right) \text{ with } v_\alpha(\mathbf{x}\,;\xi_1) = \frac{(u(\mathbf{x}\,;\xi_1, \cdot), L_\alpha)_{L^2(\Gamma_2)}}{||L_\alpha||^2_{L^2(\Gamma_2)}} \\
&= \sum_{\alpha=0}^{p_\xi} \pi^{\xi_1}_{p_\xi}(v_\alpha(\mathbf{x}\,;\xi_1)) L_\alpha(\xi_2) \\
&= \sum_{\alpha=0}^{p_\xi} \sum_{\beta=0}^{p_\xi} v_{\alpha\beta}(\mathbf{x}) L_\beta(\xi_1) L_\alpha(\xi_2) \text{ with } v_{\alpha\beta}(\mathbf{x}) = \frac{(v_\alpha(\mathbf{x}\,;\cdot), L_\beta)_{L^2(\Gamma_1)}}{||L_\beta||^2_{L^2(\Gamma_1)}}.
\end{aligned}
$$

For $p_\xi = 2$, we have $L_\alpha(\xi_2) = \{1, \xi_2, (3\xi_2^2 - 1)/2\}$ and $L_\beta(\xi_1) = \{1, \xi_1, (3\xi_1^2 - 1)/2\}$, meaning that the projector $\pi^{\xi_1}_{p_\xi} \circ \pi^{\xi_2}_{p_\xi}$ coincides with the projection onto the subspace spanned by the two-dimensional second-order PC basis, i.e., span$\{1, \xi_1, \xi_2, \xi_1\xi_2, (3\xi_1^2 - 1)/2, (3\xi_2^2 - 1)/2\}$.

Returning back to the error estimation, we consider for clarity $M = 2$ in the remaining part of the proof, that is, $\Gamma = \Gamma_1 \times \Gamma_2$. Let us estimate

$$||u - v_{p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = ||u - \left(\pi_{p_\xi}^{\xi_1} \circ \pi_{p_\xi}^{\xi_2}\right)(u)||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}.$$

We have

$$\begin{aligned}
||u - \left(\pi_{p_\xi}^{\xi_1} \circ \pi_{p_\xi}^{\xi_2}\right)(u)||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} &\leq ||u - \pi_{p_\xi}^{\xi_1} u||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + ||\pi_{p_\xi}^{\xi_1} u - \left(\pi_{p_\xi}^{\xi_1} \circ \pi_{p_\xi}^{\xi_2}\right)(u)||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \\
&\leq \Lambda p_\xi^{-k} ||u||_{L^2(\mathcal{D})\otimes H^k(\Gamma_1)\otimes L^2(\Gamma_2)} + \underbrace{||\pi_{p_\xi}^{\xi_1}||}_{\leq 1} \; ||u - \pi_{p_\xi}^{\xi_2} u||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \\
&\leq \Lambda p_\xi^{-k} ||u||_{L^2(\mathcal{D})\otimes H^k(\Gamma_1)\otimes L^2(\Gamma_2)} + \Lambda p_\xi^{-k} ||u||_{L^2(\mathcal{D})\otimes L^2(\Gamma_1)\otimes H^k(\Gamma_2)}. \quad \text{(B.1)}
\end{aligned}$$

To proceed further, we use the following result (see Lemma 2.1 in [41]). For all $k \geq 0$ and $0 \leq r \leq k$,

$$||u||_{L^2(\mathcal{D})\otimes H^r(\Gamma_1)\otimes H^{k-r}(\Gamma_2)} \leq ||u||_{L^2(\mathcal{D})\otimes H^k(\Gamma_1\times\Gamma_2)}. \quad \text{(B.2)}$$

Indeed, we have:

$$\begin{aligned}
||u||^2_{L^2(\mathcal{D})\otimes H^r(\Gamma_1)\otimes H^{k-r}(\Gamma_2)} &= \int_{\Gamma_2} \sum_{l=0}^{k-r} \left|\left| \frac{\partial^l}{\partial \xi_2^l} u(\cdot\,;\cdot,\xi_2) \right|\right|^2_{L^2(\mathcal{D})\otimes H^r(\Gamma_1)} \rho_2(\xi_2) d\xi_2 \\
&= \int_{\Gamma_2} \sum_{l=0}^{k-r} \int_{\Gamma_1} \sum_{l'=0}^{r} \left|\left| \frac{\partial^{l+l'}}{\partial \xi_1^{l'} \partial \xi_2^l} u(\cdot\,;\xi_1,\xi_2) \right|\right|^2_{L^2(\mathcal{D})} \underbrace{\rho_1(\xi_1)\rho_2(\xi_2) d\xi_1 d\xi_2}_{\rho(\boldsymbol{\xi})d\boldsymbol{\xi}} \\
&\leq \int_{\Gamma_1\times\Gamma_2} \sum_{l+l'=0}^{k} \left|\left| \frac{\partial^{l+l'}}{\partial \xi_1^{l'} \partial \xi_2^l} u(\cdot\,;\xi_1,\xi_2) \right|\right|^2_{L^2(\mathcal{D})} \rho(\boldsymbol{\xi}) d\boldsymbol{\xi} = ||u||^2_{L^2(\mathcal{D})\otimes H^k(\Gamma_1\times\Gamma_2)}.
\end{aligned}$$

Substituting (B.2) in (B.1) with $r = k$ and $r = 0$, finally gives

$$||u - v_{p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = ||u - \left(\pi_{p_\xi}^{\xi_1} \circ \pi_{p_\xi}^{\xi_2}\right)(u)||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq 2\Lambda p_\xi^{-k} ||u||_{L^2(\mathcal{D})\otimes H^k(\Gamma)},$$

that is, the error estimate (19) with $C = 2\Lambda$. The general case follows by induction which gives $C = C(M) = M\Lambda$. It is to be noted that the constant C grows linearly with M.

APPENDIX C. STABILITY ANALYSIS OF θ-WEIGHTED TEMPORAL DISCRETIZATION SCHEME

APPENDIX C.1 Proof of Lemma 2

Writing

$$\left(\frac{u_{h,p_\xi}^{m+1} - u_{h,p_\xi}^m}{\Delta t}, u_{h,p_\xi}^{m+\theta} \right)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + A(u_{h,p_\xi}^{m+\theta}, u_{h,p_\xi}^{m+\theta}) = (f_{h,p_\xi}^{m+\theta}, u_{h,p_\xi}^{m+\theta})_{L^2(\mathcal{D})\otimes L^2(\Gamma)},$$

using the equality

$$u_{h,p_\xi}^{m+\theta} = \Delta t \left(\theta - \frac{1}{2}\right) \frac{u_{h,p_\xi}^{m+1} - u_{h,p_\xi}^m}{\Delta t} + \frac{u_{h,p_\xi}^{m+1} + u_{h,p_\xi}^m}{2},$$

and using the α_e-ellipticity of A on $\mathcal{V}^h \otimes \mathcal{S}^{p_\xi} \subset \mathcal{V} \otimes \mathcal{S}$, we have

$$\begin{aligned}
\Delta t \left(\theta - \frac{1}{2}\right) \left|\left| \frac{u_{h,p_\xi}^{m+1} - u_{h,p_\xi}^m}{\Delta t} \right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} &+ \frac{1}{2\Delta t} (u_{h,p_\xi}^{m+1} - u_{h,p_\xi}^m, u_{h,p_\xi}^{m+1} + u_{h,p_\xi}^m)_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \\
&+ \alpha_e \, ||u_{h,p_\xi}^{m+\theta}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq ||f_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} ||u_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \quad \text{(C.1)}
\end{aligned}$$

For every $w_{h,p_\xi} = \sum_{|\boldsymbol{\alpha}|\le p_\xi} w_{\boldsymbol{\alpha},h}(\mathbf{x})L_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}$, we have

$$||w_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} = \sum_{|\boldsymbol{\alpha}|\le p_\xi} ||w_{\boldsymbol{\alpha},h}||^2_{L^2(\mathcal{D})}||L_{\boldsymbol{\alpha}}||^2_{L^2(\Gamma)} \le \sum_{|\boldsymbol{\alpha}|\le p_\xi} ||w_{\boldsymbol{\alpha},h}||^2_{H^1(\mathcal{D})}||L_{\boldsymbol{\alpha}}||^2_{L^2(\Gamma)} = ||w_{h,p_\xi}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)}.$$

Hence, we can use the inequality $||u^{m+\theta}_{h,p_\xi}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \ge ||u^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$. Applying the inequality $2ab \le (a^2/\epsilon) + \epsilon b^2$ $(a, b \ge 0,\ \epsilon > 0)$ with $\epsilon = \alpha_e > 0$ and using the fact that $\theta - 1/2 \ge 0$, it follows that

$$\frac{1}{2\Delta t}\left(||u^{m+1}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} - ||u^{m}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\right) + \alpha_e\, ||u^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$$

$$\le \frac{1}{2}\left(\frac{1}{\alpha_e}||f^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \alpha_e||u^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\right).$$

Reordering terms, we get

$$||u^{m+1}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} - ||u^{m}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \alpha_e\Delta t\, ||u^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \le \frac{\Delta t}{\alpha_e}\, ||f^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)},$$

leading to

$$||u^{m+1}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \le ||u^{m}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{\Delta t}{\alpha_e}\, ||f^{m+\theta}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}.$$

We then deduce the stability result (33) by induction.

APPENDIX C.2 Proof of Lemma 3

Coming back to (C.1), we have

$$\frac{1}{2\Delta t}\left(||u^{m+1}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} - ||u^{m}_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\right) + \alpha_e\, ||u^{m+\theta}_{h,p_\xi}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)}$$

$$\le \Delta t\left(\frac{1}{2} - \theta\right)\left|\left|\frac{u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi}}{\Delta t}\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + ||f^{m+\theta}_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\, ||u^{m+\theta}_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \tag{C.2}$$

In order to estimate $||(u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi})/\Delta t||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$, we substitute $v_{h,p_\xi} = (u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi})/\Delta t$ in (29) which yields

$$\left|\left|\frac{u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi}}{\Delta t}\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \le ||f^{m+\theta}_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\left|\left|\frac{u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi}}{\Delta t}\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)} - A\left(u^{m+\theta}_{h,p_\xi}, \frac{u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi}}{\Delta t}\right)$$

and hence

$$\left|\left|\frac{u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi}}{\Delta t}\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \le ||f^{m+\theta}_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\left|\left|\frac{u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi}}{\Delta t}\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$$

$$+\ \alpha_c\, ||u^{m+\theta}_{h,p_\xi}||_{H^1(D)\otimes L^2(\Gamma)}\left|\left|\frac{u^{m+1}_{h,p_\xi} - u^{m}_{h,p_\xi}}{\Delta t}\right|\right|_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \tag{C.3}$$

using the inequality (5). To proceed further, we invoke the following discrete inverse inequality that holds for quasi-uniform meshes in $\mathbb{R}^d$ (see [31], Theorem 4.5.11):

$$||w_h||_{H^1(\mathcal{D})} \le (C_i^*/h)\, ||w_h||_{L^2(\mathcal{D})},\ \forall w_h \in \mathcal{V}^h, \tag{C.4}$$

where C_i^* is a constant independent of h. Hence it follows that

$$||w_{h,p_\xi}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)} \leq (C_i^*/h)\,||w_{h,p_\xi}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)},\ \forall w_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}.$$

Applying the previous inequality to $(u_{h,p_\xi}^{m+1} - u_{h,p_\xi}^m)/\Delta t$ in (C.3), we deduce

$$\left|\left|\frac{u_{h,p_\xi}^{m+1} - u_{h,p_\xi}^m}{\Delta t}\right|\right|_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq ||f_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{\alpha_c C_i^*}{h}||u_{h,p_\xi}^{m+\theta}||_{H^1(\mathcal{D})\otimes L^2(\Gamma)}.$$

Squaring the previous inequality and applying $(a+b)^2 \leq (1+\epsilon)a^2 + [1+(1/\epsilon)]b^2$ ($a, b \geq 0$, $\epsilon > 0$), we get:

$$\left|\left|\frac{u_{h,p_\xi}^{m+1} - u_{h,p_\xi}^m}{\Delta t}\right|\right|^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq (1+\epsilon)\left(\frac{\alpha_c C_i^*}{h}\right)^2 ||u_{h,p_\xi}^{m+\theta}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} + \left(1+\frac{1}{\epsilon}\right)||f_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \quad \text{(C.5)}$$

Next, substituting (C.5) in (C.2), we obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(||u_{h,p_\xi}^{m+1}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} - ||u_{h,p_\xi}^{m}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\right)\\
&+\left(\alpha_e - \Delta t\left(\frac{1}{2}-\theta\right)(1+\epsilon)\left(\frac{\alpha_c C_i^*}{h}\right)^2\right)||u_{h,p_\xi}^{m+\theta}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)}\\
&\leq \Delta t\left(\frac{1}{2}-\theta\right)\left(1+\frac{1}{\epsilon}\right)||f_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + ||f_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\,||u_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \qquad \text{(C.6)}
\end{aligned}$$

Using the inequality $2ab \leq \gamma a^2 + \frac{1}{\gamma}b^2$ ($a, b \geq 0$, $\gamma > 0$) with $\gamma = 4\epsilon^2$, we obtain

$$||f_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\,||u_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq 2\epsilon^2||u_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{1}{8\epsilon^2}||f_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \quad \text{(C.7)}$$

For estimating $||u_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$ we use the following Poincaré's inequality

$$||w_h||_{L^2(\mathcal{D})} \leq C_{\mathcal{D}}||w_h||_{H_0^1(\mathcal{D})},\ \forall w_h \in \mathcal{V}^h \subset H_0^1(\mathcal{D}), \quad \text{(C.8)}$$

from which it follows that

$$||w_h||^2_{L^2(\mathcal{D})} \leq \frac{C_{\mathcal{D}}^2}{C_{\mathcal{D}}^2+1}||w_h||^2_{H^1(\mathcal{D})},\ \forall w_h \in \mathcal{V}^h$$

and

$$||w_{h,p_\xi}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq \frac{C_{\mathcal{D}}^2}{C_{\mathcal{D}}^2+1}||w_{h,p_\xi}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)},\ \forall w_{h,p_\xi} \in \mathcal{V}^h \otimes \mathcal{S}^{p_\xi}.$$

Hence we obtain the inequality

$$||f_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\,||u_{h,p_\xi}^{m+\theta}||_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq 2\epsilon^2\frac{C_{\mathcal{D}}^2}{C_{\mathcal{D}}^2+1}||u_{h,p_\xi}^{m+\theta}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)} + \frac{1}{8\epsilon^2}||f_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$$

that we substitute in (C.6) to get

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(||u_{h,p_\xi}^{m+1}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} - ||u_{h,p_\xi}^{m}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}\right)\\
&+\left(\alpha_e - \frac{\Delta t(1-2\theta)(1+\epsilon)\alpha_c^2(C_i^*)^2}{2h^2} - \frac{2\epsilon^2 C_{\mathcal{D}}^2}{C_{\mathcal{D}}^2+1}\right)||u_{h,p_\xi}^{m+\theta}||^2_{H^1(\mathcal{D})\otimes L^2(\Gamma)}\\
&\leq \frac{\Delta t}{2}(1-2\theta)\left(1+\frac{1}{\epsilon}\right)||f_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \frac{1}{8\epsilon^2}||f_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}. \qquad \text{(C.9)}
\end{aligned}$$

To ensure stability in (C.9), we require the following sufficient condition $\alpha_e - \beta\Delta t - \lambda \geq 0$ with

$$\beta = \frac{(1-2\theta)(1+\epsilon)\alpha_c^2(C_i^*)^2}{2h^2} > 0 \text{ and } \lambda = \frac{2\epsilon^2 C_{\mathcal{D}}^2}{C_{\mathcal{D}}^2+1} > 0.$$

Next, assuming that $\alpha_e - \lambda \geq 0$, that is,

$$0 < \epsilon \leq \left(\frac{\alpha_e(C_{\mathcal{D}}^2+1)}{2C_{\mathcal{D}}^2}\right)^{1/2},$$

it follows that

$$0 \leq \Delta t \leq \frac{\alpha_e - \lambda}{\beta} = \frac{2h^2\left(\alpha_e(C_{\mathcal{D}}^2+1) - 2\epsilon^2 C_{\mathcal{D}}^2\right)}{(C_{\mathcal{D}}^2+1)(1-2\theta)\alpha_c^2(C_i^*)^2(1+\epsilon)}.$$

Assuming that the preceding inequalities hold, we finally get

$$||u_{h,p_\xi}^{m+1}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} \leq ||u_{h,p_\xi}^{m}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)} + \Delta t\left((1-2\theta)\left(1+\frac{1}{\epsilon}\right)\Delta t + \frac{1}{4\epsilon^2}\right)||f_{h,p_\xi}^{m+\theta}||^2_{L^2(\mathcal{D})\otimes L^2(\Gamma)}$$

from which we deduce the final result (34) by induction.